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Optimal Borrowing and Developmental Planning

By

Vijay Laxman Kelkar

B.Eng. (University of Poona) 1962
M.S. (University of Minnesota) 1965

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

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in

Business Administration

in the

GRADUATE DIVISION

of the

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A dissertation is only a part of one's intellectual development. I will always cherish the intellectually exciting community of the university where I learned to learn.

CHAPTER I

THE ROLE OF EXTERNAL RESOURCES IN ECONOMIC DEVELOPMENT

There are several approaches in the literature of development economics to studying the process of economic development. These approaches are both descriptive and suggestive. One can summarize these approaches in three basic classifications: 1) The historical approach, 2) The empirical-structural approach, 3) The analytical approach. These are rather broad definitions, and may overlap.

The historical approach. This approach looks at the process of economic growth in currently industrialized countries of the European and North American continents, breaking down the economic growth process into different stages of growth. The major articulation of this view is that of W. Rostow [32]. Rostow sees economic development traversing through the following stages: 1) traditional society, 2) pre-conditions for take-off, 3) take-off, 4) self-sustained growth, 5) high mass consumption. The central stage is, of course, the "take-off" stage. This stage is characterized by social change, increased investment (5-10%), and the technical transformation of the production process. This creates the growth of the leading sector, which is usually the export-generating sector (e.g., cotton in the United States, silk in Japan, etc.). These exports plus the capital imports from abroad generate the supply of investment goods, enabling the economy to move towards sustained growth.

This description is by no means accepted by all economists. S. Kuznets [18] devastatingly attacks it on empirical grounds. He argues that the stages observed were not empirically similar and, indeed, the sequence was not so rigid. Professor A. Gerschenkron [10] has interesting observations on the process of industrialization. He emphasizes the uniqueness of the response of each country to the process of development, and sees this uniqueness as depending upon the degree of backwardness of the country. The more backward the country, the higher the degree of technical change and the more important are capital goods industries.

Professor Gerschenkron is much influenced by his study of the Russian experience. But the limitations of such historical studies are obvious. Even though they may be sufficiently descriptive of past experience, they cannot be applied to present day underdeveloped countries. In fact, the more useful analysis would be to answer the question of why the present day underdeveloped countries are backward.

In our judgment the important work in this direction is that of Professor Paul Baran [3]. Professor Baran stresses the importance of the colonial nature of those countries which were exploited by the metropolitan powers, with the result that economic growth was stifled. He argues that even present day international economic relations between developed and underdeveloped countries lead to the same results.

<u>The structural and empirical approach</u>. This approach revolves around the detailed empirical examination of the developed countries and then deduces the similarities and the differences of the process of economic growth. Professor S. Kuznets remarks that there are a variety of forces at work. Essentially, he sees economic growth as the interacting process between the demand for and the supply of productive forces. Population growth, natural resources, and technical change affect the process of

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Professor Chenery [6] emphasizes the structural interdependence of the economy. The developing economy has to increase the flow of investible funds, with the balance of payments as the constraint. This external balance constraint is more effective in the later stage. In the earlier stage, the investment demand is larger than the supply of domestic savings. This dynamic disequilibrium is replaced by equilibrium through the effective use of external resources (e.g., foreign aid).

Professors Bruno and Chenery [7] exploit activity analysis techniques to characterize the solution to the micro-dynamic problems of what to produce and how much to produce suggested by the macro-dynamic analysis just described.

The analytical approach. This approach constructs descriptive models of underdeveloped economies and then suggests policy measures (controls) to facilitate economic development. Professor Leibenstein [19] argues that the underdeveloped economy is a low-level equilibrium system. The growth-generating forces are counteracted by the growth-dampening forces (i.e., demographic forces). His analysis is essentially neo-Malthusian, and his way out is through what he calls "critical minimum efforts"--"quantum jumps" in investment. Such investments break the low-level equilibrium and put the economy on the path of accelerated growth. The source of these large scale investments are the external resources injected into the domestic economy. Such an increase in investment, of course, requires a supply of complementary factors like technically skilled manpower and organizational skills. In fact, the paucity of complementary factors creates the problem of "absorptive capacity"--there is a maximum level of investment that can be absorbed by the domestic economy.

Another analytic model is the celebrated model of the dual economy, first suggested by Sir Arthur Lewis [20]. He sees the underdeveloped economy as consisting of two sectors: the traditional sector (agriculture) and the modern sector (manufacturing). Another feature of the dual economy is the existence of surplus labor. The constraints on growth in this model are the supply of marketable surplus and the size of the investment goods sector. The supply of external resources alleviates these constraints. External resources can either increase the marketable surplus (e.g., the supply of consumer goods), thus increasing employment in the manufacturing sector, or it can increase the stock of capital goods in the manufacturing sector, which increases the capacity to generate investment goods.

Thus, in all the models two elements are important: 1) the process of capital accumulation and 2) the supply and the use of external resources in the financing of capital accumulation. To put it compactly, the problem is one of transforming the internal resources into external resources to achieve the desired level of capital stock. This transformation can be achieved through the adjustment of flows (trade) or the adjustment of stocks (international capital movements).

This transformation of resources is the crux of the matter. The traditional trade theorist emphasizes, from static equilibrium considerations, the role of free trade and export industries. He argues that free trade was, after all, the engine of growth in the 19th century. However, this source of transformation is not feasible today to the extent that it was a century ago. In this age there are many factors in international economic relations working against the underdeveloped economies.

The source of major export earnings of underdeveloped countries are primary commodities [i.e., raw materials, mineral ores, agricultural goods and semi-manufactured goods (fibers, textiles, etc.)], while the imports of these countries are essentially capital goods and industrial intermediate goods. The following are the major difficulties in attempting to increase the export earnings of the underdeveloped economies: 1) The demand for these goods is income inelastic; as income in the developed countries goes up, their demand for these goods does not increase at as high a rate as income increases. 2) The demand for goods exported by underdeveloped countries is a derived demand. As technology in the developed countries changes, the technology develops synthetic substitutes for the primary commodities it has traditionally imported. In this way, the total demand for traditional imports is reduced. 3) There is a secular trend of falling prices for primary commodities vis-a-vis the prices of investment goods (i.e., there is deterioration of terms of trade). 4) Most importantly, the trade policies of the developed countries are discriminatory in their tariff structures. Tariffs are directed against semi-processed agricultural goods and industrial raw materials. Effective tariff rates (computed in the sense of value added in the corresponding rates of developed countries) are quite high, as the following Table 1 shows.

Table 1*

Nominal and Effective Protection Rates
in the U.S. and European Economic Community

Processing industry	U.S. Rates		E.E.C. Rates	
	Nominal	Effective	Nominal	Effective
Coconut oil	5.7	57.5	15.0	15.0
Fabrics	3.1	5.3	23.0	39.6
Cigarettes	47.1	89.0	N.A.	N.A.
Hand fiber mfg.	15.1	38.0	N.A.	N.A.

*Source: H. Johnson, "Trade Policies toward Less Developed Countries," p. 91, Table 2, The Brookings Institution, Washington, 1967.

The average effective tariff rate on intermediate and consumer goods industries in the developed countries is 25%. The surplus disposal of certain U.S. agricultural commodities has a similar adverse effect on the possibilities of export expansion for the underdeveloped economies. The United Nations Conference on Trade and Development (UNCTAD) has the following estimates on the loss of export earnings of underdeveloped countries. The following figures are quoted from Professor Johnson's [14] study.

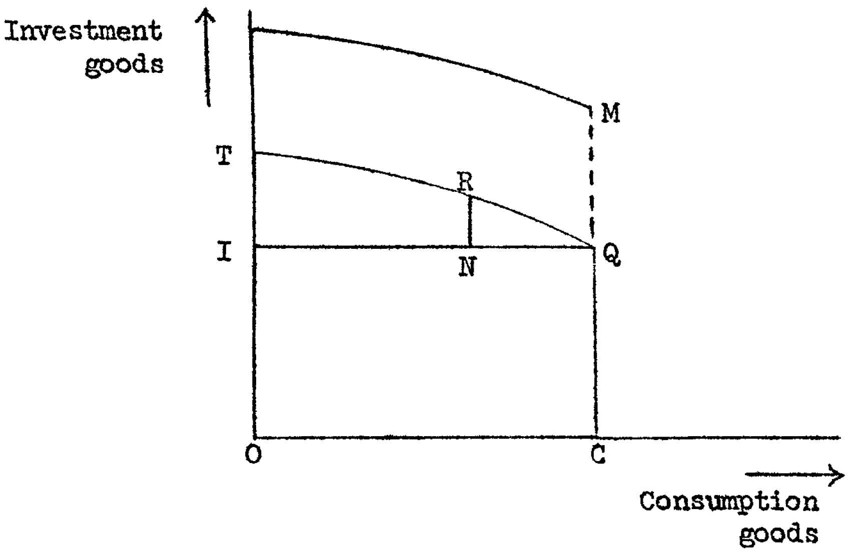
agricultural protectionism in developed countries	$2000 million
sugar protection	$ 400 million
European duties and excise on coffee, tea, bananas, etc.	$ 125 million
U.S. surplus disposal	$ 685 million

There is a systematic non-tariff and tariff discrimination against semi-manufactured goods by the developed economies.

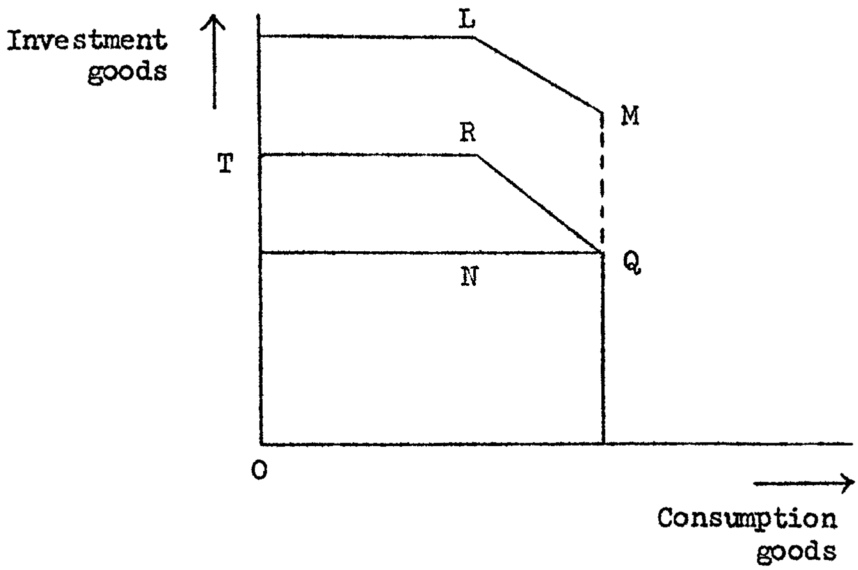
In summary, the transformation of domestic resources into external resources to finance needed capital and industrial goods for investment through trade is increasingly difficult. This phenomenon is described by Professor Nurkse [25], who points out that the export as an engine of growth is now in "comparatively low gear." The case for the industrialization of the primary commodity-producing countries which are oriented primarily towards home markets has been recognized on these grounds.

The external resources generated by borrowing, compared with those generated by trade, are different in one crucial way. Foreign borrowing increases the total supply of resources for expenditure, along with the supply of needed imports. Trade changes the composition of the expenditure, but not necessarily the expenditure level. In this sense, the alternative of borrowing or aid is more appealing to underdeveloped economies, especially when their imports are essential for capital accumulation. This important conceptual difference can best be illustrated by the following simple model. Consider an economy which produces only one good, namely corn, and employs tractors and corn as inputs in the production of the corn. The only way the economy can augment its capital stock of tractors is through trade, i.e., by buying tractors from abroad in exchange for its product, namely corn. In this way the economy either reduces domestic consumption in order to import tractors or it reduces the domestic investment. Or the economy can borrow funds from abroad (which increases the production of corn) and then repay the funds borrowed. This strategy is still more attractive if the transformation of corn into tractors is increasingly more expensive in terms of corn, i.e., domestic resources. The following diagram illustrates the point.

Let Q be the initial stock mix of the domestic economy (OC = consumption goods and OI = investment goods). Let the economy be willing to save QN for exports, which are transformed into RN, investment goods, where QRT is the locus of transformation from consumption goods into investment goods.



The above diagram shows such a transformation. If, on the other hand, QM is external borrowing, the locus of transformation is shifted upwards by the distance QM, and the economy may well choose borrowing in preference to exports in the short run. In the long run it is not obvious what should be the choice. Of course, if the exports are strictly constrained as to absolute level, then even if there are no declining terms of trade (as shown in the following diagram), the effect is the same.



Either the difficulty of extracting enough domestic resources or the loss involved in transforming domestic resources into external resources, or (generally) the two together may, and almost universally do, prevent the country from achieving a growth rate adequate to its economic objective. From this situation is derived the potential macro-economic contributions, respectively, of external resources and external trade.

Today most underdeveloped countries depend heavily on external resources to increase their per capita income. A crude indicator of this dependence is the net flow of about 10 billion dollars per year from advanced countries to less developed countries, an amount which is equal to one quarter of their gross investment and nearly one third of their imports [8]. Thus, these external resources both augment domestic savings and provide many goods which are strategically important in efficient economic growth. These goods mainly comprise heavy machinery and machine tools, some industrial raw materials, and fuels.

The problem of allocation of foreign borrowing can be analyzed on two levels. One level, which I call the aggregate level, involves deciding the volume of this stock transaction (either borrowing or repayment or lending) and its timing. The other level involves the choice of the sectoral allocation of resources, i.e., the decision whether to invest the resources in machines that produce the machines or to invest them in machines that produce consumer goods (e.g., textile machinery, such as manufacturing tolls, etc.). I will discuss each of these problems in the following chapters. It is obvious that the two levels are interdependent. However, it is helpful for the planner to pose the question in terms of different levels and to treat the two different aspects of the problem separately.

The main work in the area of developmental planning on an aggregate level is that of Professor Chenery and other AID economists [9], who work with the Harrod-Domar type one-sector aggregate model, which generates the flow of external resources needed by an underdeveloped economy. These resources are in the nature of aid so the problems of repayment and debt servicing are not considered. The role of external resources is to fill the dual gaps of imports and exports and savings and investment.

A model of this sort is as follows:

1. $M_t + V_t = I_t + E_t + C_t$
2. $V_t = C_t + S_t$
3. $M_t = F_t + E_t$
4. $I_t = S_t + F_t$
5. $\Delta K_t = I_t$
6. $I_t - I_{t-1} \leq \beta I_{t-1}$
7. $E_t = E_0(1 + \varepsilon)^t$
8. $\bar{S}_t = (a_0 - a')V_0 + a'V_t$
9. $\bar{M}_t = (u_0 - u')V_0 + u'V_t$
10. $K_t = V_t k$

Symbol	Description
V_t	G.N.P.
I_t	investment
K_t	capital stock
E_t	exports
F_t	aid
$\bar{S}_t, S_t$	intended and realized savings
r	target growth rate
k	capital-output ratio
$\bar{M}_t, M_t$	intended and realized imports

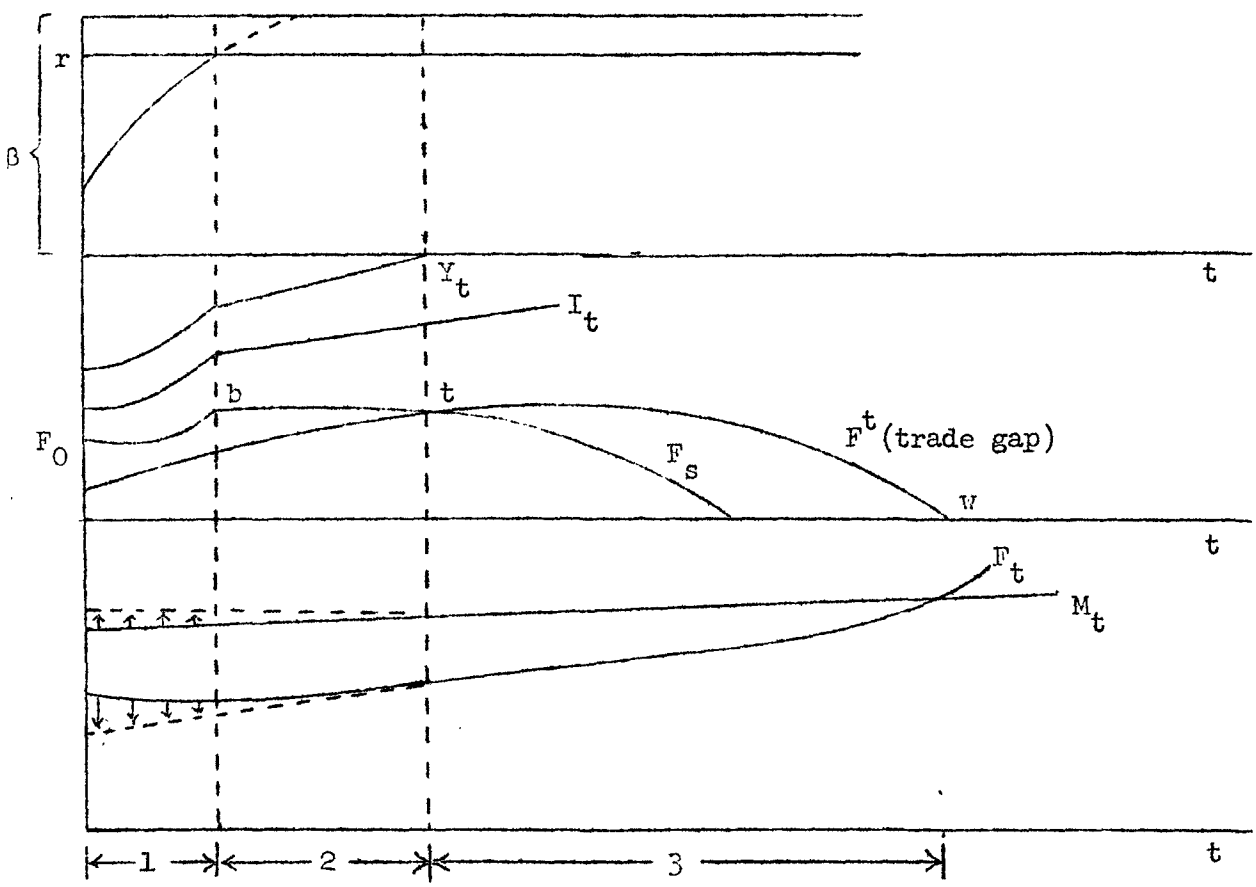
The behavioral and technological assumptions are the following:

1. there is a maximum marginal savings rate (a');
2. a constant capital-output ratio (k);
3. a constant marginal import propensity (u');
4. an exogenous rate of export growth (ε);
5. an absorptive capacity coefficient (β).

We note that the Equations (8) and (9) give the intended savings and imports. The levels actually realized may be different than those intended. Of course, ex post, the value of F_t calculated from Equation (3) is the same as that calculated from Equation (4); the ex ante values may be different.

The model generates the growth path shown in the following figure. There are three phases in the growth pattern.

Phase I: skill-limited growth
Phase II: savings-limited growth
Phase III: trade-limited growth



In Phase I the absorptive capacity (the skills formation required of managers, skilled labor and civil servants in order to make productive investments) is the limit and the investment meets this constraint, i.e., $I_t = (1 + \beta)I_{t-1}$. This phase continues until the growth rate becomes equal to the target growth rate "r". In this phase the "aid" required is the gap between savings and investment.

In Phase II the income growth rate is "r", i.e., the target growth rate. From the constant capital-output ratio one can determine the required investment rate and the difference between savings and investment gives the needed "aid".

In Phase III the growth rate is r and the required imports are determined by Equation (9) while the exports are rising at the constant rate ε. The "aid" is equal to the trade gap. In summary,

In Phase I $\quad \text{aid} = F^s = (F + \dot{M}(1 + \beta)^t) - \dot{M}$

$$\text{where } \dot{M} = (\alpha_0 - \alpha')V_0 + \frac{\alpha(K_0 - I_0/\beta)}{k}$$

In Phase II $\quad F^s = \text{aid} = (k_r - \alpha') \cdot V_0(1 + r)^t - (\alpha - \alpha')V_0$

In Phase III $\quad F^t = \text{aid} = (u - u')V_0 + u'V_0(1 + r)^t - E_0e^{\varepsilon t}$

where F^s denotes the savings gap and F^t the trade gap.

In all the phases aid is determined by the maximum of the trade gap and the savings gap. The ex post equality of these two gaps is obvious since

$$F = \text{aid} = \max(F^s, F^t) \qquad \text{for all times } t.$$

The curve Fbtw in the diagram above is the aid path. The trade gap and savings gap close eventually, provided the following assumptions are made about the parameters.

1. $r < t < r\left(\frac{u'V_0}{E_0}\right)$

2. $u' > E_0/V_0$

3. $I_0/K_0 < \beta$

4. $r < \beta$

The ex post equality of the two gaps means that in Phase I and II the realized imports or exports are not the same as the ex ante ones, while in Phase III actual savings are not the same as ex ante savings. This means that in Phase I and II imports may indeed be more than what is needed or exports less than the economy's ability to export, while in Phase III potential savings are not realized. As Professors Ranis and Fei [30] point out, this is the basic weakness of the AID.analysis. The point becomes more vivid when we know that all these external resources have a "cost" attached to them, i.e., they have to be repaid with interest in the future. The analysis assumes that the country borrows a maximum of the two gaps. With a borrowing cost introduced, the country should adjust its growth rate so that the debt burden is optimal. The terms at which money for imports can be borrowed crucially effect the level of borrowing. Professor Quayyam [28] has done some calculations showing that if loans carry 4 per cent interest and a specified repayment period, then under certain conditions on productivity the costs of these external loans will outweigh the benefits in the long run. In fact, he cautions against indiscriminate acceptance of external aid. In such instances myopic borrowing policies may become non-optimal. For this reason the AID approach to determining the flow of external resources is unsatisfactory from an underdeveloped country's viewpoint.

The second class of models which have been suggested to study the flow of external resources is of the neo-classical variety. Bardhan [3] studies this problem in the single-sector growth model context. He assumes that a country can borrow funds from abroad at increasing rates of interest. Then he finds a golden-rule accumulation path which gives the optimal borrowing path. In his maximand, however, he has a curious assumption about the disutility from the existing debt level. This assumption is not convincing. But the main limitation in applying Bardham's model in the underdeveloped economy is that it neglects the important role of the imported capital goods.

Imported capital goods are qualitatively different from domestic capital goods, and they are a sine-qua-non of efficient industrialization. Bardhan also has in his model a closed economy, i.e., no exports are available to finance these needs of the economy. Hamada [12] constructs a similar model.

The following points summarize the drawbacks of the models of allocation of external resources that have been suggested so far:

1. The AID model falls short of being truly dynamic in the way it handles optimal savings and investment allocation.
2. The AID model fails to recognize that external resources have "costs" attached to them that must be repaid.
3. The neoclassical one-sector model misses the essential importance of external capital goods because it assumes a closed economy.

In Chapter II we formulate the problem in the neoclassical growth model framework. This is essentially a one-sector aggregate macro-model.

We will describe the structure of our model and study its simple dynamic properties. In Chapter III we will solve the optimization problem in the context of our model. In that analysis we will study the qualitative nature of the optimal controls, and in doing so we shall answer the question of the level of external borrowing, its timing and its repayment. In Chapter IV data on the Indian economy will be used to estimate the uncontrolled parameters of the model and a computer program will then be used to study optimal paths when the uncontrolled parameters are set at these levels.

CHAPTER II

ANALYSIS OF THE MODEL WITHOUT OPTIMIZATION

Description of the model. The developing economy exhibits a distinct qualitative difference from an economy which is mature in that the developing economy cannot produce all types of investment goods. This may be due to lack of natural resources, technology and skilled manpower. To produce machines one requires other machines and these machines have to be imported. Similarly, the economy requires certain industrial raw materials and fuels (gasoline, kerosene, etc.) which may not be locally available and therefore have to be imported. This would not be a problem if conversion of domestic resources into external resources was easy. This transformation is feasible via international trade (i.e., exports-imports). Thus transformation or conversion depends upon the exports, which may be either stagnant, as Professors Raj and Sen hypothesize [29], or else the exports may be linked to the level of industrialization of the underdeveloped economy, as Professor R. McKinnon [23] suggests.

I will make the assumption that the maximum possible export earnings depend upon the level of industrialization. Export earnings present such difficulties as these: 1) advanced countries' tariff barriers against the industrial or semi-industrial goods which can be exported by the underdeveloped countries (these barriers are lessened, however, if the goods are technologically more sophisticated); 2) low income elasticity of demand for underdeveloped countries' exports; 3) the increased use of synthetic substitutes in the developed countries. Then the way out of these constraints is to borrow the external resources. This borrowing makes available the resources but in so doing it incurs costs, namely, interest payments and the obligation to pay back the debt in the future.

Thus, there is trade-off between present gains and future costs or obligations. This is an economic problem, and the development planner has to address himself to the problem of the timing and the amount of borrowing.

In the model of the economy to be studied, I will assume that the economy is centrally planned. The planner has the knowledge of society's preference ordering (i.e., social welfare function) and production function. The planner guarantees full employment. The economy produces one good which is either consumed, invested or exported. The rule of imported goods is two-fold. They furnish raw materials, spare parts and fuels to maintain the current economic activity (i.e., maintenance imports) and they also provide complementary capital goods in the accumulation of capital for future economic activity (i.e., they increase the capital stock). Full utilization of the capital stock is also assumed. The planner controls the savings as well as the exports in the economy. This presumably can be done via appropriate fiscal and monetary means. The exports are constrained from above. The economy can borrow the resources from abroad with a given interest rate schedule and has to repay through its exports earnings. The planner uses a constant discount rate to compare different streams of social utility and chooses the program which maximizes the discounted sum of this utility stream. The units are chosen so that one unit of domestic goods can buy a unit of imported goods. The international prices are assumed to remain constant. The growth of the population is exponential (the growth rate is constant), and is exogeneously determined. The elements of the model are as follows, where the variables are understood to be functions of time.

(a) Production function

$$Y = F(K,L)$$

where K = capital stock, L = labor, and Y = output.

The production function has constant returns to scale and exhibits the usual neoclassical properties, i.e.,

(1) $\lambda Y = F(\lambda K, \lambda L)$

Using (1) we may write

$$y = \frac{Y}{L} = \frac{1}{L} F(K/L, 1) = f(k)$$

where $k = \frac{K}{L}$, the capital-labor ratio.

(2) $f'(k) > 0, \; f''(k) > 0, \quad$ for all k, $0 < k < \infty$.

We assume the following end-point conditions of the Inada type:

(3) $\lim\limits_{k \to \infty} f(k) = \infty , \qquad \lim\limits_{k \to 0} f(k) = 0$

$\lim\limits_{k \to \infty} f'(k) = 0 \qquad \lim\limits_{k \to 0} f'(k) = \infty$

(b) Social utility function

The domain of this function is per capita consumption and is assumed to be strictly concave.

Let $c = \frac{C}{L}$, where C = consumption and c = per capita consumption Then social utility function is denoted by u(c)

where

$$u'(c) > 0, \; u''(c) < 0 \quad \text{for all } 0 < c < \infty$$

and

$$\lim_{c \to 0} u'(c) = \infty .$$

(c) Population growth

$\dot{L} = nL$ where n is a constant and is exogeneously given (a dot over a variable denotes its derivative with respect to time).

(d) Capital accumulation

The rate of change of net per capita capital stock is equal to the total per capita output minus the interest payments, per capita consumption and depreciation:

(4) $\dot{k} = [f(k)(1 - v - \sigma) - r(d) \cdot d](1 + \beta) - (\mu + n)k,$

where β, σ, μ, v are defined below.

(e) External sector

The change in the per capita debt is equal to the total per capita imports minus total exports and minus or times debt per capita. Imports per capita are

$$[f(k)(1 - v - \sigma) - r(d) \cdot d]\beta + \quad f(k),$$

exports per capita are

$$\sigma f(k),$$

and

$$\dot{d} = [f(k)(1 - v - \sigma) - r(d) \cdot d]\beta + \alpha f(k) - \sigma f(k) - nd,$$

where in (d) and (e)

β = number of units of externally produced capital goods per unit of gross domestic investment

μ = the depreciation rate

$vf(k)$ = per capita consumption

$\sigma f(k)$ = per capita exports

d = per capita debt

$r(d) \cdot d$ = interest payments on the outstanding debt d

$\alpha f(k)$ = current imports of raw materials and intermediate goods

We assume that both β and α are constant and that:

$\sigma \leqq \lambda$, where λ is the maximum propensity to export

$0 \leqq \alpha \leqq \lambda, \quad \beta > 0$

$v + \sigma \leqq 1.$

(f) Debt repayment

We assume

$$r(d) > 0, \qquad r'(d) > 0, \qquad r''(d) > 0 \qquad \text{if } d > 0$$

$$r(d) > 0, \qquad r'(d) < 0, \qquad r''(d) < 0 \qquad \text{if } d < 0,$$

where $R(d) = r(d) \cdot d$ is the per capita service charge on the debt.

(g) Objective function

The economy is to maximize the sum of the discounted utility stream subject to all the above constraints.

$$\text{Maximize} \quad \int_0^\infty e^{-\delta t}\, u(v \cdot f(k))dt$$

$$\text{Subject to} \quad d \geqq 0 \text{ and } (a)\text{-}(f)$$

In the optimization problem the planner treats 'v' and 'σ' as the control variables; "δ" is the discount rate reflecting the time preference of the planner in evaluating a program.

Comparative dynamics of the model. In order to better understand the working of this Model I will present some simple dynamic properties of the model specified by (a)-(f) before studying the solution of the optimization problem which is obtained when (g) is added. To do so I shall assume for simplicity that control parameters are set so as to achieve constant average propensity to consume and constant average propensity to export.

That is to say,

$$\frac{\text{per capita consumption}}{\text{per capita output}} = \bar{v} \text{ (i.e., for all points of time } t, \; v = \bar{v})$$

$$\frac{\text{per capita exports}}{\text{per capita output}} = \bar{\sigma} \text{ (i.e., for all } t, \; \sigma = \bar{\sigma})$$

where

$$\bar{v} + \bar{\sigma} \leq 1 \qquad \bar{\sigma} \leq \lambda$$

The permitted values of the control parameters are shown diagrammatically in the figure below.

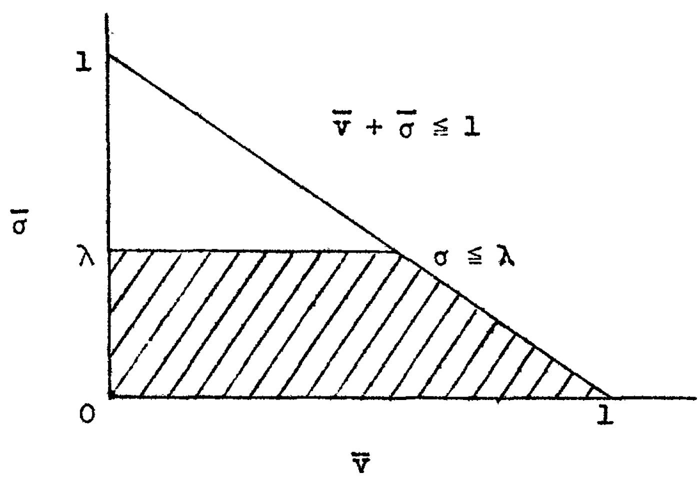


Figure 1

From (d) and (e) we have

$$\dot{k} = [f(k)(1 - \bar{v} - \bar{\sigma}) - r(d) \cdot d](1 + \beta) - (\mu + n)k$$

$$\dot{d} = [f(k)(1 - \bar{v} - \bar{\sigma}) - r(d) \cdot d]\beta + \alpha f(k) - \bar{\sigma} f(k) - nd.$$

The movement of k and d will be different in the different cases depending upon the interest schedule and the values of parameters. We shall consider two possible interest schedules corresponding to the case of perfect international capital market and the case of imperfect international capita markets.

(i) Competitive international capital market

This occurs when the funds can be borrowed or lent at the market price and the country is small enough so that it is a price-taker.

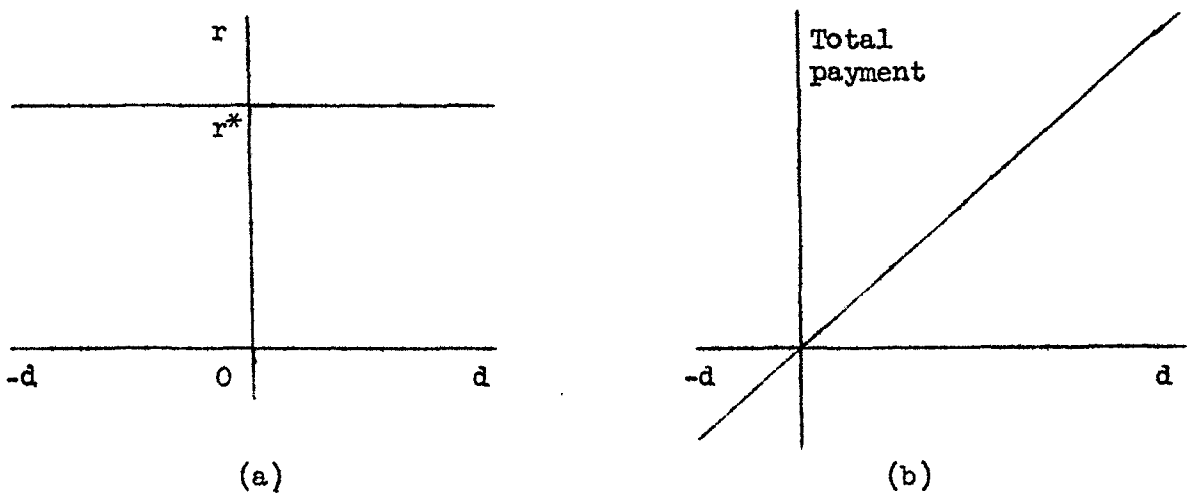


Figure 2

The above figure describes the interest payments under (i); $d > 0$ means debt and $d < 0$ means lending.

(ii) Imperfect capital market

This is characterized by the fact that borrowing is increasingly more costly, i.e., the marginal interest payments exceed average interest payments and lending is increasingly less renumerative, i.e., the marginal return is less than the average return.

To be precise, we have

$$r(d) > 0 \qquad \bar{d} < d < \infty \qquad\qquad r(d) = 0 \qquad d < \bar{d}$$

$$r'(d) > 0 \qquad d > 0 \qquad\qquad r'(d) < 0 \qquad d < 0$$

The marginal interest payments $R'(d)$ thus depends upon $r'(d)$. The following figure (3) is illustrative.

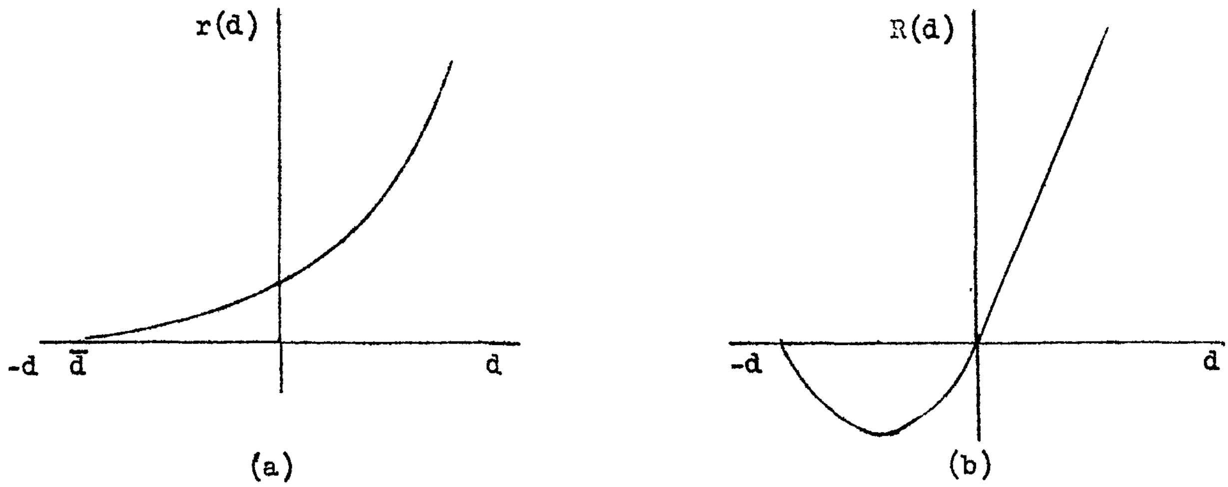


Figure 3

It is clear that the case of a perfectly competitive capital market is a special case of an imperfect capital market such that $r'(d) = 0$ and $\bar{d} = \infty$. Now we have from Equation (4)

$$\dot{k} \left\{\begin{matrix}>\\<\end{matrix}\right\} 0 = f(k)(1 - \bar{v} - \bar{\sigma})(1 + \beta) - (\mu + n)k \left\{\begin{matrix}>\\<\end{matrix}\right\} r(d) \cdot d(1 + \beta)$$

To simplify the algebra, assume that

$$1 - \bar{v} - \bar{\sigma} = a.$$

Then

$$\dot{k} \left\{\begin{matrix}>\\<\end{matrix}\right\} 0 = f(k)a(1 + \beta) - (\mu + n)k \left\{\begin{matrix}>\\<\end{matrix}\right\} r(d)d(1 + \beta)$$

The situation is depicted in Figure 4. Let $\hat{k}$ denote the maximum capital/output ratio sustainable without borrowing. Then

$$f(\hat{k})a(1 + \beta) = (\mu + n)\hat{k}$$

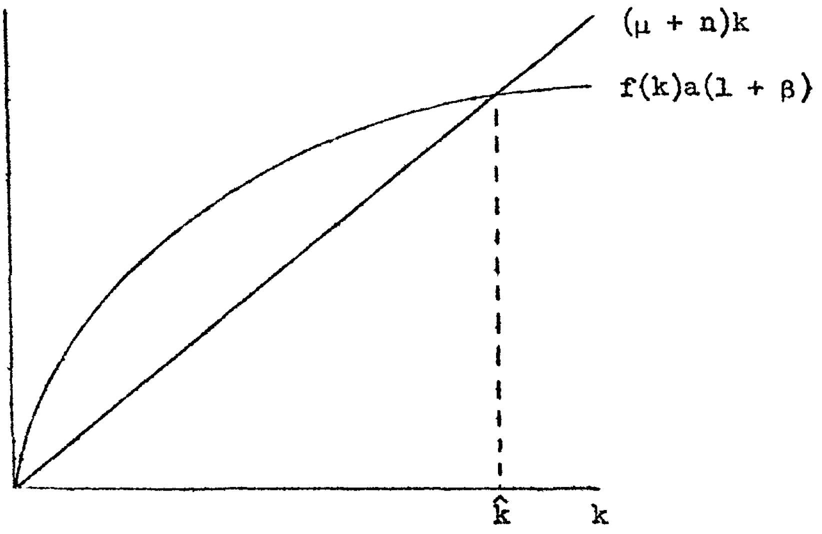


Figure 4

It is then obvious from the following diagram that if, at $\bar{k}$, $d > 0$ and $\dot{k} = 0$, then $\bar{k}$ will be less than $\hat{k}$.

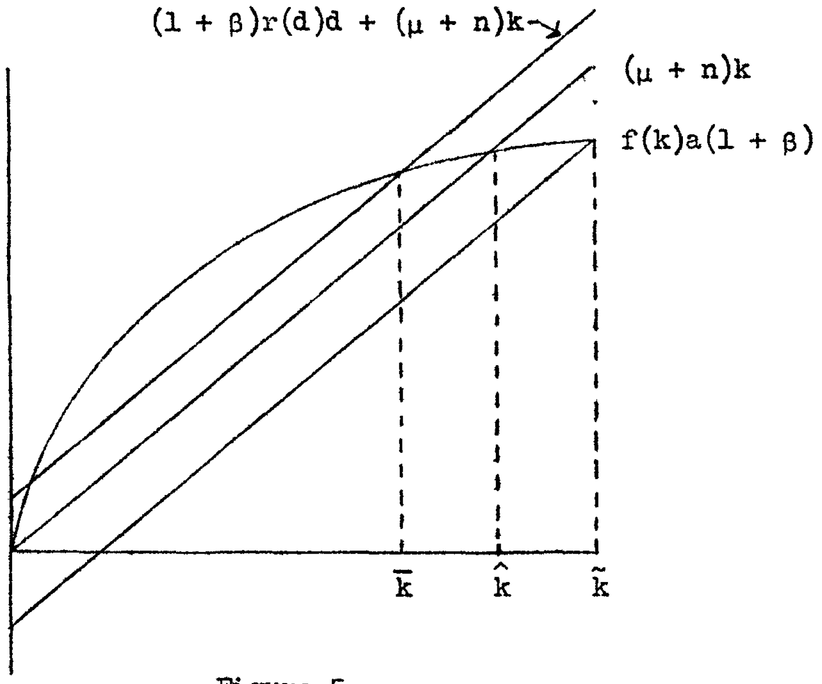


Figure 5

Also, if, at $\tilde{k}$, $\bar{d} < d < 0$ and $\dot{k} = 0$, then $\tilde{k}$ will be greater than $\hat{k}$.

Then the following diagram gives the locus of the curve $\dot{k} = 0$ in (k,d) space.

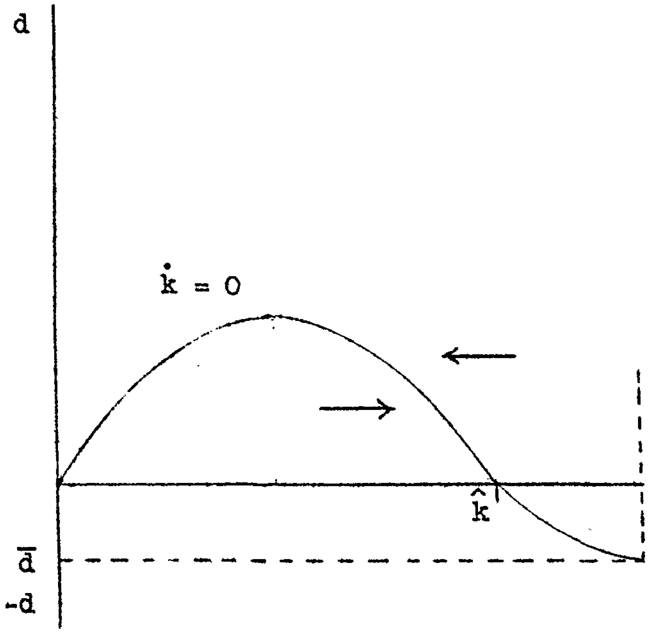


Figure 6

This locus is determined by the implicit differentiation of the curve $\dot{k} = 0$, i.e., by the values taken by the expression

$$\left.\frac{dd}{dk}\right|_{\dot{k}=0} = \frac{f'(k)a(1 + \beta) - (\mu + n)}{R'(d)} ,$$

which determines the nature of the curve $\dot{k} = 0$. Arrows show the movement of k in that region of the (k,d) space.

Now let us consider the movement of debt. We have

$$\dot{d} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \Rightarrow f(k)(a\beta + \alpha - \sigma) \begin{Bmatrix} > \\ < \end{Bmatrix} [r(d)\beta + n]d.$$

Two cases need to be distinguished:

Case (i) $a\beta + \alpha - \bar{\sigma} > 0$

Case (ii) $a\beta + \alpha - \bar{\sigma} < 0$

In Case (i) the imports always exceed the exports, which means that $\dot{d} = 0$ is possible if and only if d is negative, i.e., if the excess imports are financed by the interest earnings from abroad. In Case (ii) the exports always exceed the imports, which implies that $\dot{d} = 0$ is feasible if and only if d is positive, i.e., the excess exports are absorbed in repaying the interest payments. Thus, in Case (i) there exists no point in (k,d) space such that $\dot{d} = 0$ and $d > 0$ and in Case (ii) there exists no point in (k,d) space such that $\dot{d} = 0$ and $d < 0$.

The following diagram summarizes the discussion.

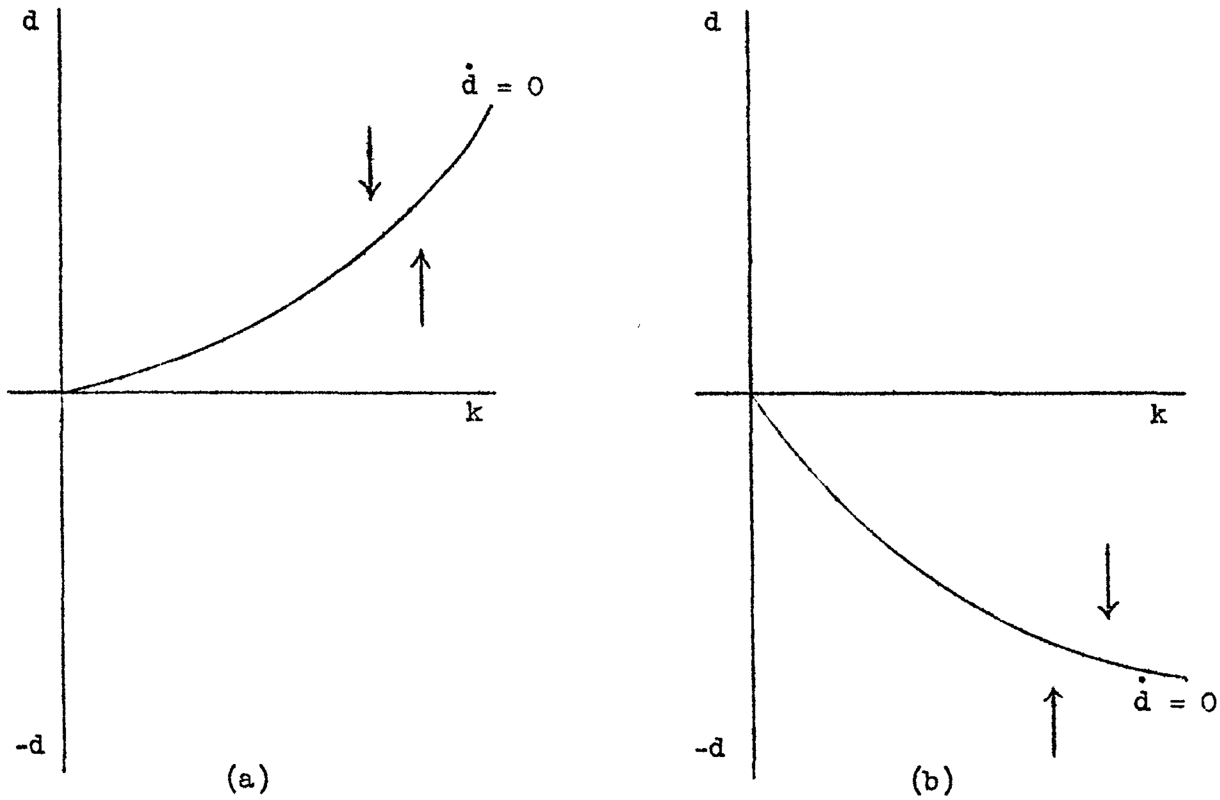


Figure 7

$$\left.\frac{dd}{dk}\right|_{\dot{d}=0} = \frac{f'(k)(a\beta + \alpha - \bar{\sigma})}{r'(d)\beta d + r(d)\beta + n}$$

and again the arrows depict the movement of d in the regions of (k,d) space.

We will examine the motion of the system under the assumption of imperfect capital market, since, as we have remarked, the perfectly competitive capital market is a special case.

Movement of economy in Case (i). We have noticed that under Case (i) $a\beta + \alpha - \bar{\sigma} > 0$, i.e., $d < 0$ on the locus $\dot{d} = 0$. We are interested in the stability of the system. In the economic sense we are interested in the stationary state--the state in which $\dot{k} = 0$ and $\dot{d} = 0$--and we shall study its properties.

The following diagram is illustrative.

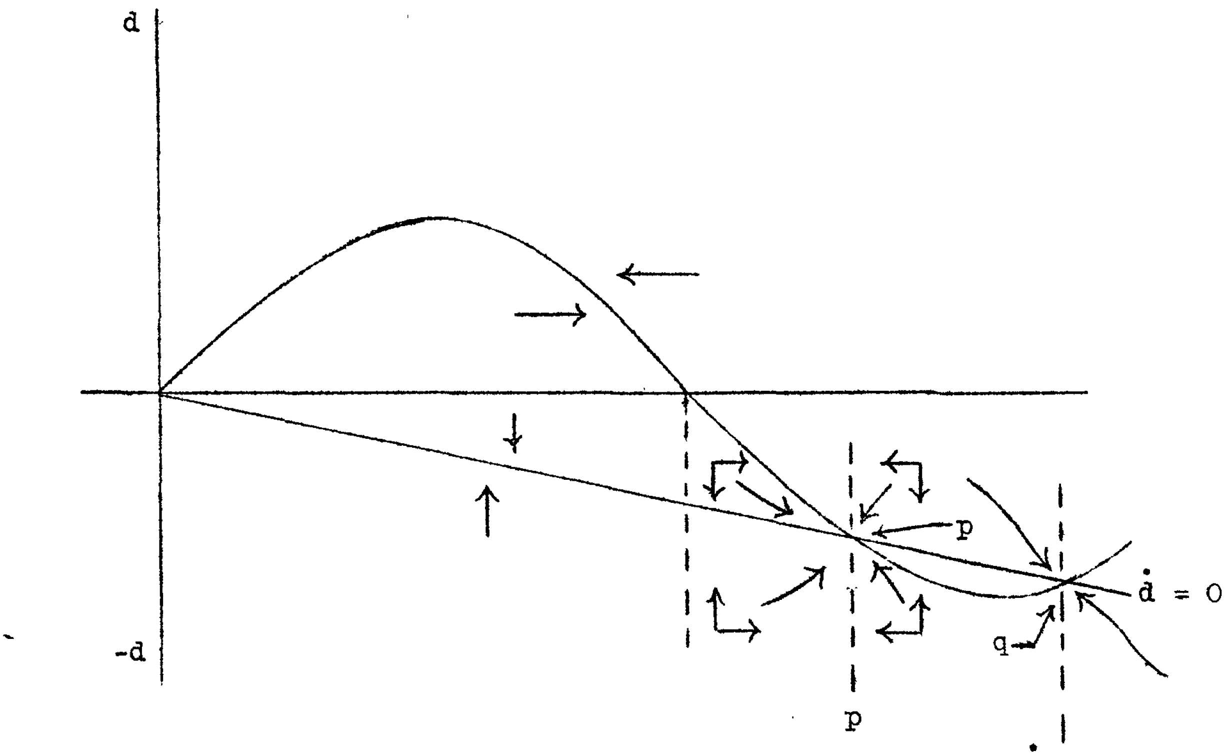
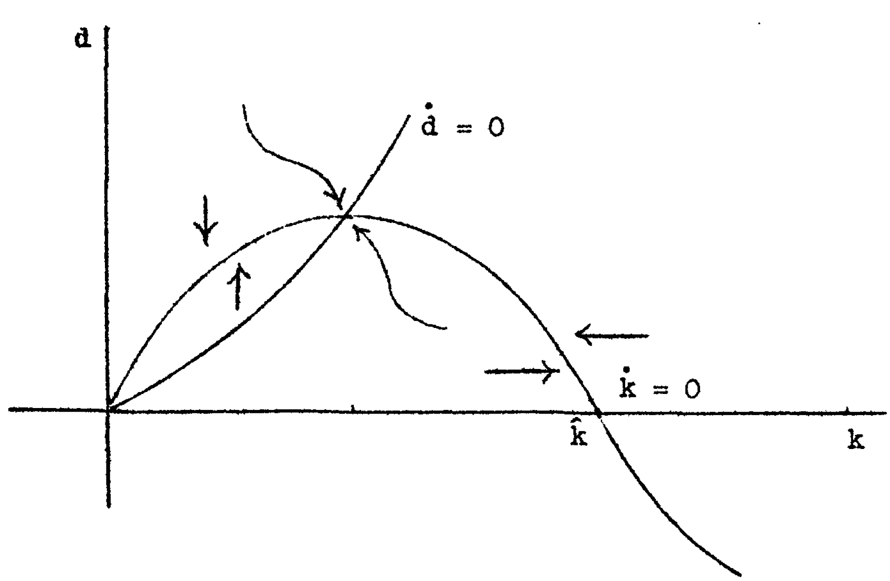


Figure 8

Clearly, the stationary point is the intersection of the curves $\dot{k} = 0$ and $\dot{d} = 0$. The following are the possibilities: i) there is no intersection, i.e., there is no stationary point; (ii) there may be only one intersection which may be stable or unstable; (iii) there may be two intersection points as shown in Figure 8. In the latter case 'p' is a stable point, i.e., the economy asymptotically grows towards p, while 'q' is a saddle point, i.e., there exists a unique trajectory going through 'q', and if the economy is initially on this trajectory, then it asymptotically grows towards 'q'.

<u>Movement of the economy in Case (ii)</u>. We have noted that under Case (ii) $a\beta + \alpha - \overline{\sigma} < 0$, i.e., $d > 0$ on the locus $\dot{d} = 0$. The stability analysis shows that if the stationary point exists in this case then it is a saddle point q, i.e., if the economy is initially on the trajectory passing through q, then the economy grows asymptotically to a stationary state. If the economy is not initially on this trajectory then there is no equilibrium point; the economy moves in such a way that the debt level grows initially and then both (k and d) eventually go to the origin, i.e., the economy decays.

The following diagram is illustrative.



The stability analysis just concluded can be presented in a compact way if we can study the two differential equations system around the stationary point (k^*, d^*) if such a point exists. We linearize the system around (k^*, d^*) by taking the linear terms of Taylor's expansion.

$$\dot{k} = a_{11}k + a_{12}d$$

$$\dot{d} = a_{21}k + a_{22}d$$

Let λ_1 and λ_2 be the eigen-values of the matrix $A = (a_{ij})$. These eigen-values describe the nature of motion of the system. If

i) $\lambda_1, \lambda_2 < 0$, the system is stable;

ii) $\lambda_1, \lambda_2 > 0$, the system is unstable;

iii) one of the pair λ_1, λ_2 has a negative real part while the other has a positive real part, then the stationary point is a saddle point;

iv) λ_1 and λ_2 are imaginary, then the system has cycles.

Thus, we get a variety of situations depending upon the initial state and the control parameters. The nature of equilibrium depends on the interest rate schedule.

More interesting theoretically in this model is the analysis of the golden rule path to which we now turn. This is a sort of "Gedanken-experiment". The question it answers, in the context of our model, is the following: what would be the optimal-capital-output ratio and per-capita consumption if the economy were required to choose a per-capita capital endowment and to maintain it indefinitely. We want to choose such a k that it maximizes the per-capita consumption. In the economic growth literature this capital-labor ratio is called the 'golden-rule' capital-output ratio. The problem is

$$\text{Maximize} \quad c = vf(k)$$

$$\text{Subject to} \quad 0 = \dot{k} = [f(k)(1 - v - \sigma) - r(d)d](1 + \beta) - (\mu + n)k$$

$$0 = \dot{d} = [f(k)(1 - v - \sigma) - r(d)d]\beta + \alpha f(k) - \sigma f(k) - nd.$$

To study this contrained maximization problem we set up a suitable Lagrangean function, namely,

$$L = vf(k) + \lambda_1\{[f(k)(1 - v - \sigma) - r(d)d](1 + \beta) - (\mu + n)k\} + \lambda_2[f(k)(1 - v - \sigma) - r(d)d]\beta + \alpha f(k) - \sigma f(k) - nd]$$

The critical point of this Lagrangean gives the necessary conditions. We get

$$\frac{\partial L}{\partial v} = [1 - \lambda_1(1 + \beta) - \lambda_2\beta]f(k) = 0$$

$$\frac{\partial L}{\partial \sigma} = (\lambda_1 + \lambda_2)(1 + \beta)f(k) = 0$$

$$\frac{\partial L}{\partial k} = [vf'(k) + \lambda_1(1 + \beta)(1 - v - \sigma)f'(k) - \mu + n] + \lambda_2[1 - v - \sigma)\beta f'(k) + \alpha f'(k) - \sigma f'(k)] = 0,$$

which implies

$$vf'(k)[1 - \lambda_1(1 + \beta) - \lambda_2\beta] + \sigma f'(k)(1 + \beta)(\lambda_1 + \lambda_2) + [\lambda_1(1 + \beta) + \lambda_2\beta - \alpha] f'(k) - (\mu + n) = 0.$$

But $\frac{\partial L}{\partial v} = \frac{\partial L}{\partial \sigma} = 0$, $\frac{\partial L}{\partial k} = 0$ and hence

$$f'(k)(1 - \alpha) = \mu + n.$$

Since $f(k)$ is strictly concave there exists a k^* such that

$$f'(k^*)(1 - \alpha) = \mu + n.$$

Also

$$\frac{\partial L}{\partial v} = \frac{\partial L}{\partial \sigma} = 0 \text{ implies } |\lambda_1| = |\lambda_2| = 1 \text{ and } \lambda_1 + \lambda_2 = 0.$$

which implies

$$R'(d) = n$$

if there exists a d^* such that $R'(d^*) = n$; otherwise, we have a corner solution. Thus, after calculating k^*, d^* we can find the values of v and σ, say v^* and σ^*, such that $\dot{k} = \dot{d} = 0$.

The optimal capital-output ratio is, then, such that net marginal productivity is equal to the population growth rate plus the depreciation rate. The per capita debt is such that it equates the marginal interest to the rate of population growth. The effect of the coefficient β is on v^* and σ^* rather than on k^*. The effect of the coefficient α is on k^* and, in turn, on all the variables. If $\alpha = 0$ we get the familiar value of k^* discussed by Koopmans [16].

This discussion naturally leads to our main problem. As we have shown, there is a variety of patterns of growth depending on the initial conditions, the savings rate $(1 - \overline{v})$, and the rate of exports $(\overline{\sigma})$. But there is no reason why any economy should have constancy of savings and exports ratio. Instead, especially for an underdeveloped economy the planner should choose a path which is optimal in some sense and this is very unlikely to be a path with constant savings and exports ratios.

CHAPTER III

THE SOLUTION OF THE OPTIMAL PROGRAM

The criterion for optimality in our model is of the Ramsey-Fisher type. This reduces the problem of optimal intertemporal allocation to the problem of ordering different programs in an ordinal sense and choosing the one which achieves an extremum.

The technique we shall use is that of Pontryagin's Maximum Principle [26]. In addition, certain results of Arrow's related work [1] will be used. We shall first state in general terms the necessary and sufficient conditions for an extremum in Arrow's notation and shall then apply it to our model. All variables are functions of time.

Let

$$x = \text{a state vector}$$

$$v = \text{a control or decision vector}$$

Let the state vector be required to satisfy the transition relations

$$\dot{x} = T(x,v,t)$$

and the constraints

$$F(x,v,t) > 0.$$

The problem is to

$$\text{Maximize} \quad \int_0^\infty e^{-\delta(t)} u(x,v,t)dt$$

subject to these conditions.

Define a real valued function, called the current-valued Hamiltonian, as follows

$$H = \hat{H}e^{\delta t} = u(x,v,t) + pT(x,v,t) + qF(x,v,t).$$

The variable p is a "co-state vector" and is a continuous function of time; in a number of contexts it may be interpreted as a price vector. Define the Lagrangean expression

$$L = H + qF(x,v,t)$$

Assume that

the functions $u(x,v,t)$, $T(x,v,t)$, $F(x,v,t)$ are independent of time (stationarity);

the Hamiltonian is a concave function of x for given p and t.

General propositions. We shall use two propositions, which can be established.

Proposition 1. Let v^* be a choice of instrument which maximizes the objective functional. Then there exists co-state variables, continuous functions of time $p(t)$, such that for each t

1. v^* maximizes the Hamiltonian H subject to the constraints $F(x,v) \geq 0$
2. $\dot{p}_i = \delta p_i - \frac{\partial L}{\partial x_i}$ and

   the Lagrangean multipliers q are such that

$$\frac{\partial L}{\partial v_k} = 0, \qquad q \geq 0, \qquad qF(x,v) = 0$$

3. the p_i satisfy the following transversality condition:

$$\lim_{t \to \infty} e^{-\delta t} p_i(t) = 0, \qquad \text{for all } i$$

Proposition 1 gives necessary conditions for a maximum while the following proposition gives sufficient conditions.

Proposition 2. Let (x, v^*, p) be the Pontryagin path (the path which fulfills the necessary conditions stated in Proposition 1). Then if the assumptions of stationarity and concavity of the Hamiltonian hold, and if along this path x and p converge to a stationary state (x^*, p^*)--values of x and p for which $\dot{x} = \dot{p} = 0$--and if $p^* \geqq 0$, then the path is an optimal path.

Application to the model. We can now apply these propositions to our model. In our case we have

$$H = u[vf(k)] + p_1\dot{k} + p_2\dot{d}$$

$$\text{and } L = H + q\dot{d}$$

Then v^* and σ^* maximize the Hamiltonian and

$$\dot{p}_1 = \delta p_1 - \frac{\partial L}{\partial K}$$

$$\dot{p}_2 = \delta p_2 - \frac{\partial L}{\partial d}$$

$$q \geqq 0, \quad qd = 0, \quad \text{for all } t;$$

$$\lim_{t \to \infty} e^{-\delta t} p_i(t) = 0 \qquad i = 1,2.$$

The economic interpretation of the co-state variables is as follows: they measure the marginal contribution of the corresponding state variables to the utility functional. They are "dynamic" Lagrangean multipliers, or social "shadow prices". This means that the Hamiltonian is the current flow of utility from all sources (both the current utility and that anticipated in the future). The future contribution is given by the inner product of p_i and x_i. The negative sign of p_2 occurs because the debt d is a burden on the economy so that its contribution is negative. The equations concerning $\dot{p}$ say that with perfect foresight the sum of capital gains and the marginal productivity of the capital should be equal to the interest payments.

Now we have for the Lagrangean expression

$$L = u[vf(k)] + p_1\{[f(k)(1 - v - \sigma) - r(d)d](1 + \beta) - (\mu + n)k\}$$
$$+ (p_2 + q)\{[f(k)(1 - v - \sigma) - r(d)d]\beta + \alpha f(k) - \sigma f(k) - nd\}.$$

Maximization of L with respect to v, σ gives the following necessary conditions. First,

$$\frac{\partial L}{\partial v} \leq 0 \text{ ,i.e., } \frac{\partial L}{\partial v} = [u' - p_1(1 + \beta) - (p_2 + q)\beta]f(k) \leq 0.$$

This implies

$$u' = p_1(1 + \beta) + (p_2 + q)\beta \text{ and v is an interior point, i.e., } 0 < v < 1$$

or

$$u' > p_1(1 + \beta) + (p_2 + q)\beta, \qquad \text{and} \qquad v = 1$$

or

$$u' < p_1(1 + \beta) + (p_2 + q)\beta \qquad \text{and} \qquad v = 0.$$

Clearly, since $u'(c) \to \infty$ as $c \to 0$, v will never become zero in an optimal program.

Secondly,

$$\frac{\partial L}{\partial \sigma} \leq 0; \frac{\partial L}{\partial \sigma} = -(p_1 + p_2 + q)(1 + \beta)f(k) \leq 0.$$

This implies

$$p_1 + p_2 + q = 0, \qquad 0 < \sigma < \lambda$$

or

$$p_1 + p_2 + q < 0, \qquad \sigma = \lambda$$

or

$$p_1 + p_2 + q > 0, \qquad \sigma = 0$$

and from Proposition 1

$$\begin{aligned} \dot{p}_1 = \delta p_1 - \frac{\partial L}{\partial k} = (\delta + \mu + n)p_1 &- u'\{v[f(k)]\}v'[f(k)]f'(k) \\ &- p_1(1 - v - \sigma)(1 + \beta)f'(k) \\ &- (p_2 + q)[(1 - v - \sigma)\beta + \alpha - \sigma]f'(k) \end{aligned}$$

and

$$\dot{p}_2 = \delta p_2 - \frac{\partial L}{\partial d} = (\delta + n)p_2 + [p_1(1 + \beta) + (p_2 + q)\beta][r'(d)d + r(d)] .$$

Thus, different modes of motion are possible for the economy, depending upon the choice of r and σ. The following table summarizes the twelve possible modes. They are divided into three "patterns" and four "cases" and are derived from the above conditions on $\frac{\partial L}{\partial v}$, $\frac{\partial L}{\partial \sigma}$ and on d (i.e., the condition $d = 0$ or $d > 0$).

These twelve distinct modes of motion are described fully by the corresponding system of four autonomous differential equations. We will look for the description of a stationary state in each mode, that is, a long-run equilibrium for the economy. From Proposition 2 we know that the path is an optimal path if the corresponding vector p is non-negative and there is convergence to the stationary state. In the next section we study the modes of motion and investigate the fulfillment of these conditions.

	Pattern I $\sigma = 0$	Pattern II $\sigma = \lambda$	Pattern III $0 < \sigma < \lambda$
Case (a) $v + \sigma = 1$ $d > 0$	$u' > p_1(1+\beta) + p_2\beta$ $p_1 + p_2 > 0$	$u' > p_1(1+\beta) + p_2\beta$ $p_1 + p_2 < 0$	$u' > p_1$ $p_1 + p_2 = 0$
Case (b) $v + \sigma = 1$ $d = 0$	$u' > p_1(1+\beta) + (p_2 + q)\beta$ $p_1 + p_2 + q > 0$	$u' > p_1(1+\beta) + (p_2 + q)\beta$ $p_1 + p_2 + q < 0$	$u' > p_1$ $p_1 + p_2 + q = 0$
Case (c) $0 < v < 1$ $d > 0$	$u' = p_1(1+\beta) + p_2\beta$ $p_1 + p_2 > 0$	$u' = p_1(1+\beta) + p_2\beta$ $p_1 + p_2 < 0$	$u' = p_1$ $p_1 + p_2 = 0$
Case (d) $0 < v < 1$ $d = 0$	$u' = p_1(1+\beta) + (p_2 + q)\beta$ $q + p_1 + p_2 > 0$	$u' = p_1(1+\beta) + (p_2 + q)\beta$ $q + p_1 + p_2 < 0$	$u' = p_1$ $p_1 + p_2 + q = 0$

Pattern I

Case (a)

$$\dot{k} = -[(\mu + n)k + r(d)d(1 + \beta)]$$

$$\dot{d} = \alpha f(k) - r(d)d\beta - nd$$

$$\dot{p}_1 = (\delta + \mu + n)p_1 - u'f'(k)p_1 - \alpha f'(k)p_2$$

$$\dot{p}_2 = (\delta + n)p_2 + [p_1(1 + \beta) + p_2\beta]R'(d).$$ Note:

$$R'(d) = r'(d)d + r(d)$$

Case (b)

$$\dot{k} = -[(\mu + n)k + r(d)d(1 + \beta)]$$

$$\dot{d} = \alpha f(k) - r(d)d\beta - nd$$

$$\dot{p}_1 = (\delta + \mu + n)p_1 - \mu' p_1 f'(k) - \alpha f'(k)(p_2 + q)$$

$$\dot{p}_2 = (\delta + n)p_2 + [p_1(1 + \beta) + (p_2 + q)\beta]R'(d)$$

Case (c)

$$\dot{k} = [f(k)(1 - v) - r(d)d](1 + \beta) - (\mu + n)k$$

$$\dot{d} = [f(k)(1 - v) - r(d)d]\beta + \alpha f(k) - nd$$

$$\dot{p}_1 = (\delta + \mu + n)p_1 - f'(k)[p_1(1 + \beta) + p_2\beta] - p_2\alpha f'(k)$$

$$\dot{p}_2 = (\delta + n)p_2 + R'(d)[p_1(1 + \beta) + p_2\beta]$$

Case (d)

$$\dot{k} = f(k)(1 - v)(1 + \beta) - (\mu + n)k$$

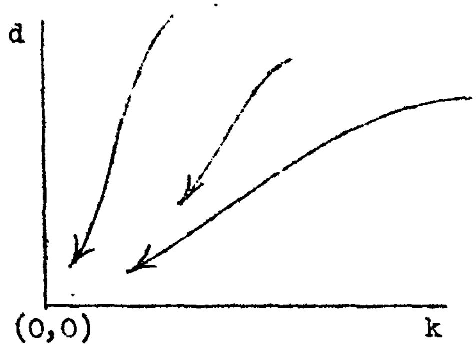
$$\dot{d} = f(k)(1 - v)\beta + \alpha f(k)$$

$$\dot{p}_1 = (\delta + \mu + n)p_1 - f'(k)[p_1(1 + \beta) + p_2\beta - p_2\alpha]$$

$$\dot{p}_2 = (\delta + n)p_2 - R'(d)[p_1(1 + \beta) + p_2\beta]$$

Cases (b) and (d) of the Pattern I economy can occur only for an instant since $\dot{d} \neq 0$, so that $d = 0$ only for an instant. In Cases (a) and (c) the stationary state is feasible if $k = d = 0$, but then p_1 and p_2 fail to satisfy the transversality condition since $p_1 \to \infty$.

The following is the phase diagram for Pattern I.



So Pattern I cannot be the terminal phase of an optimal program.

Pattern II

Case (a)

$$\dot{k} = -[(\mu + n)k + r(d)d]$$

$$\dot{d} = (\alpha - \lambda)f(k) - nd$$

$$\dot{p}_1 = (\delta + \mu + n) - f'(k)\{(1 - \lambda)[u' - p_1(1 + \beta) - p_2\beta] - (\alpha - \lambda)p_2\}$$

$$\dot{p}_2 = (\delta + n)p_2 + [p_1(1 + \beta) + p_2\beta]R'(d)$$

Case (b)

$$\dot{k} = -(\mu + n)k$$

$$\dot{d} = (\alpha - \lambda)f(k)$$

$$\dot{p}_1 = (\delta + \mu + n)p_1 - f'(k)[(1 - \lambda)(u' - p_1)(1 + \beta) - (p_2 + q)\beta - (\alpha - \lambda)(p_2 + q)]$$

$$\dot{p}_2 = (\delta + n)p_2 + [p_1(1 + \beta) + (p_2 + q)\beta]R'(d)$$

Case (c)

$$\dot{k} = [f(k)(1 - v - \lambda) - r(d)d](1 + \beta) - (\mu + n)k$$

$$\dot{d} = [f(k)(1 - v - \lambda) - r(d)d]\beta + \alpha f(k) - \lambda f(k) - nd$$

$$\dot{p}_1 = (\delta + \mu + n)p_1 + \lambda f'(k)[(p_1 + p_2)(1 + \beta)(1 - \tfrac{1}{\lambda})] - p_2\alpha f'(k)$$

$$\dot{p}_2 = (\delta + n)p_2 + [p_1(1 + \beta) + p_2\beta]R'(d)$$

Case (d)

$$\dot{k} = f(k)(1 - v - \lambda)(1 + \beta) - (\mu + n)k$$

$$\dot{d} = f(k)[(1 - v - \lambda)\beta + \alpha - \lambda]$$

$$\dot{p}_1 = (\delta + \mu + n)p_1 + f'(k)[\lambda(p_1 + p_2 + q)(1 + \beta) - p_1(1 + \beta) + \beta(p_2 + q) - (p_2 + q)\alpha]$$

$$\dot{p}_2 = (\delta + n)p_2 + [p_1(1 + \beta) + \beta(p_2 + q)R'(d)]$$

It is obvious that in this pattern Cases (b) and (d) are achieved only momentarily. This is due to the fact that these modes have d = 0 but $\dot{d} \neq 0$. In Case (a) the economy will grow to the origin asymptotically but the co-state variables violate the transversality condition while d has no stationary state.

Thus, the Pattern II cannot be the terminal phase of an optimal program.

<u>Pattern III</u>

Case (a)
$$\dot{k} = -[(\mu + n)k + r(d)d(1 + \beta)]$$
$$\dot{d} = \alpha f(k) - nd$$
$$\dot{p}_1 = (\delta + \mu + n)p_1 - f'(k)(1 - \alpha)p_1$$
$$\dot{p}_2 = [\delta + n - R'(d)]p_2$$

Case (b)
$$\dot{k} = -(\mu + n)k$$
$$\dot{d} = \alpha f(k)$$
$$\dot{p}_1 = (\delta + \mu + n)p_1 - f'(k)(1 - \alpha)p_1$$
$$\dot{p}_2 = [(\delta + n) - R'(d)]p_2$$

Case (c)
$$\dot{k} = [f(k)(1 - v - \sigma) - r(d)d](1 + \beta) - (\mu + n)k$$
$$\dot{d} = [f(k)(1 - v - \sigma) - r(d)d]\beta - \sigma f(k) + \alpha f(k) - nd$$
$$\dot{p}_1 = (\delta + \mu + n) - f'(k)(1 - \alpha)p_1$$
$$\dot{p}_2 = (\delta + n) - R'(d)p_2$$

Case (d)
$$\dot{k} = f(k)(1 - v - \sigma)(1 + \beta) - (\mu + n)k$$
$$\dot{d} = f(k)[(1 - v - \sigma)\beta + \alpha - \sigma]$$
$$\dot{p}_1 = (\delta + \mu + n)p_1 - f'(k)(1 - \alpha)p_1$$
$$\dot{p}_2 = (\delta + n)p_2 - (p_2 + q)R'(0)$$

In this pattern Case (a) has the stationary state at the origin but violates the transversality conditions, while the system can stay in Case (b) only for a moment. We are therefore left with Cases (c) and (d) in this pattern. Here a stationarity state is possible and the trajectory is in the interior of the control set. We will call this an "Euler Trajectory".

The study of the Euler Trajectory. This is a possible candidate for the terminal phase of an optimal trajectory. Its stationary point is of interest, as with the nature of the motion of the system.

To stay in this pattern of motion for a finite period of time implies

$$p_1 + p_2 = 0 \text{ and } \dot{p}_1 + \dot{p}_2 = 0$$

The equations for $\dot{p}_1$ and $\dot{p}_2$ were given above. They are

$$\dot{p}_1 = (\delta + \mu + n)p_1 - f'(k)(1 - \alpha)$$

$$\dot{p}_2 = (\delta + n)p_2 - R'(d)p_2$$

Hence, $p_1 + p_2 = 0$ and $\dot{p}_1 + \dot{p}_2 = 0$ imply

$$f'(k)(1 - \alpha) - \mu = R'(d) \tag{1}$$

The left-hand side of Equation (1) is the productivity of the capital net of depreciation, while the right-hand side is the marginal interest payment. On the Euler trajectory they are equal. This is precisely the condition for efficient atemporal allocation (marginal benefits equal marginal cost). So the level of debt satisfying (1) is the optimal level of debt for the economy. The left-hand side of (1) is a decreasing function of k (since f(k) is strictly concave) and the right-hand side is an increasing function of d. Equation (1) gives us the implicit relation between the optimal level of debt and capital/labor ratio. To study the properties of this relation we differentiate totally. This gives

$$f''(k)(1 - \alpha)\frac{dk}{dd} = r''(d)d + 2r'(d)$$

and

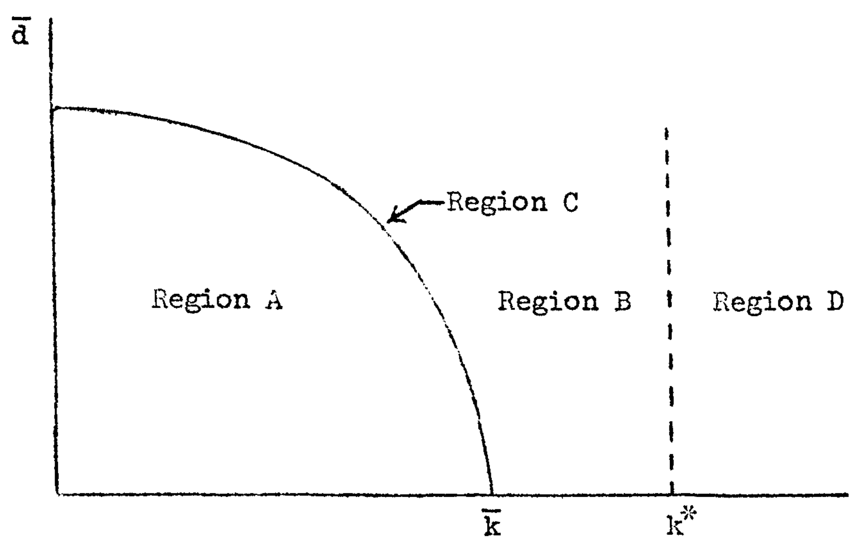
$$\frac{dd}{dk} = \frac{f''(k)(1 - \alpha)}{r''(d)d + 2r'(d)}$$

so if $\bar{d} = g(k)$ where $\bar{d}$ = optimal level of debt then

$$\left.\frac{dd}{dk}\right|_{\bar{d}=g(k)} < 0$$

since $f''(k) < 0$, $r''(d) > 0$, $r'(d) > 0$ and $(1 - \alpha) > 0$. This implies that as the capital/labor ratio increases on the Euler trajectory, per capita debt decreases.

The function g(k) is shown in the following diagram.



The function $g(k)$ creates four distinct regions in the (k,d) phase space.

1. Region A: $f'(k)(1 - \alpha) > R'(d) + \mu$
2. Region B: $f'(k)(1 - \alpha) < R'(d) + \mu$
3. Region C: $f'(d)(1 - \alpha) = R'(d) + \mu$
4. Region D: $k > k^*$, where k^* turns out to be the stationary-state capital/labor ratio.

The value $\tilde{k}$ satisfies $f'(k)(1 - \alpha) = R'(0) + \mu$.

We now turn to the study of the stationary states on the Euler trajectory. There are two distinct possible stationary states where the economy can grow. To which stationary state the economy will tend depends upon (1) the interest rate schedule and (2) the planner's discount rate δ. Both these stationary states have the same capital-labor ratio.

At the stationary state the motion of the system becomes zero, i.e.,

$$\dot{p}_1 = \dot{p}_2 = \dot{k} = \dot{d} = 0. \tag{2}$$

Let k^*, d^*, $p_1{}^*$, $p_2{}^*$ be the stationary state.

Then, solving (2) we obtain

$$f'(k^*)(1 - \alpha) = \delta + \mu + n$$

Since $f(k)$ is a strictly concave function, k^* is unique. We also obtain

$$d^* = \begin{cases} g(k^*) & \text{if } r(0) < \delta + n \\ 0 & \text{if } r(0) > \delta + n \end{cases}$$

$$p_1{}^* = -p_2{}^* = u'[v^* f(k^*)],$$

where v^* and σ^* satisfy

$$[f(k^*)(1 - v^* - \sigma^*) - r(d^*)d^*](1 + \beta) = (\mu + n)k^*$$

$$[f(k^*)(1 - v^* - \sigma^*) - r(d^*)d^*]\beta + \sigma f(k^*) - \sigma f(k^*) - nd^* = 0$$

and

$$\sigma^* \leqq \lambda .$$

Now,

$$k^* < \infty , \quad v^* < 1 ,$$

which implies

$$0 < u'[v^* f(k^*)] < \infty .$$

That means $p_1{}^* = u'[v^* f(k^*)]$ is finite and hence the transversality condition $p_i{}^* > 0$ is satisfied. Therefore,

$$\lim_{t \to \infty} p_i{}^* e^{-\delta t} = 0 \qquad i = 1,2$$

It follows from Proposition 2 that the Euler trajectory, when it exists, is the terminal phase of the optimum program.

Now we have a pair of non-linear, autonomous differential equations. To study the existence of the trajectory we study the system in the neighborhood of the stationary point. We take the Taylor's expansion around the stationary point and linearize the system. We then study the signs of the eigen-values of matrix of the resulting linear differential equation system.

Because $u'\{v[f(k)]\} = p_1$, $vf(k)$ is a function of p_1 . Then since exports $\sigma f(k)$ are a function of $vf(k)$, we can write

$$(v + \sigma)f(k) = \Pi(p_1).$$

We note that $\frac{d\Pi(p_1)}{dp} < 0$. We have

$$\dot{R} = (1 + \beta)[f(k) - \Pi(p_1)] - (\mu + n)k$$

$$\dot{p}_1 = [\delta + \mu + n - f'(k)(1 - \alpha)]p_1$$

Expanding around the stationary point we have

$$\begin{bmatrix} \dot{k} \\ \dot{p}_1 \end{bmatrix} = \begin{bmatrix} f'(k)(1 + \beta) - (\mu + n) & -\Pi'(p_1) \\ -f''(k)(1 - \alpha)p_1 & \delta + \mu + n - f'(k)(1 - \alpha) \end{bmatrix} \begin{bmatrix} k \\ p_1 \end{bmatrix}$$

The eigen-values of this matrix are λ_1 and λ_2, where λ_1 and λ_2 are the solutions of the quadratic equation

$$\lambda^2 - [f'(k)\alpha + \delta]\lambda - [\Pi'(p_1)f''(k) + f'(k) - (\mu + n)] = 0$$

Let

$$-[f'(k)\alpha + \delta] \equiv b$$

and

$$-[\Pi'(p_1)f''(k) + f'(k) - (\mu + n)] \equiv c$$

Then

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4c}}{2},$$

But

$$f'(k) > 0 \text{ implies } b > 0$$

and

$$\Pi'(p_1) < 0, \quad f''(k) < 0 \text{ implies } c < 0.$$

This means that $\lambda_1 > 0$, $\lambda_2 < 0$. Hence, the stationary point is a saddle point. Therefore, there exists a unique trajectory passing through the stationary point and this is the natural candidate for the optimal trajectory[27].

Synthesis of controls. We can now qualitatively characterize the optimal program. The aim of the optimal trajectory is to become an Euler trajectory in its terminal phase. We can see what controls the economy ought to choose to achieve this aim when it is in any of the regions A, B, C or D defined in the preceding diagram. For this purpose we prove several lemmas.

Lemma 1. If the economy is in the region C it stays in the region, choosing the Euler trajectory, and asymptotically grows to the stationary state.

Proof. We appeal to Proposition 2, since the Euler trajectory satisfies the both necessary and sufficient conditions of that proposition.

Now we turn to regions A and B. We need the following lemmas to qualitatively characterize the controls.

Lemma 2. Switching from Pattern I to Pattern III occurs only in region A.

Proof. In Pattern I, $p_1 + p_2 > 0$. So for switching from Pattern I to Pattern III to occur (Pattern III is the terminal point), in the neighborhood of the switching point

$$p_1 + p_2 = 0$$

and

$$\frac{\dot{p}_1}{\dot{p}_2} < \frac{\dot{p}_2}{p_2},$$

which means

$$(\delta + \mu + n) - f'(k)(1 - \alpha) < (\delta + n) - R'(d)$$

$$f'(k)(1 - \alpha) > R'(d) + \mu.$$

But this, by definition, is region A.

Lemma 3. Switching from Pattern II to Pattern III occurs only in region B.

Proof. The argument is similar to that used for the previous lemma. In the neighborhood of the switching point we must have

$$p_1 + p_2 = 0$$

and

$$\frac{\dot{p}_1}{p_1} > \frac{\dot{p}_2}{p_2}$$

This implies

$$f'(k)(1 - \alpha) < R'(d) + \mu,$$

which, by definition, is region B.

Lemma 4. Switching from Pattern I to II or from Pattern II to I cannot occur.

Proof. If switching from I to II were to occur in the neighborhood of the switching point the following would be true:

$$p_1 + p_2 = 0 \text{ and } \dot{p}_1 < \dot{p}_2$$

This implies

$$f'(k)(1 - \alpha) > R'(k) + \mu$$

which can only occur in region A. Similarly, if switching from II to I were to occur it would have to be in region B. The optimal trajectory has to switch to Pattern III eventually.

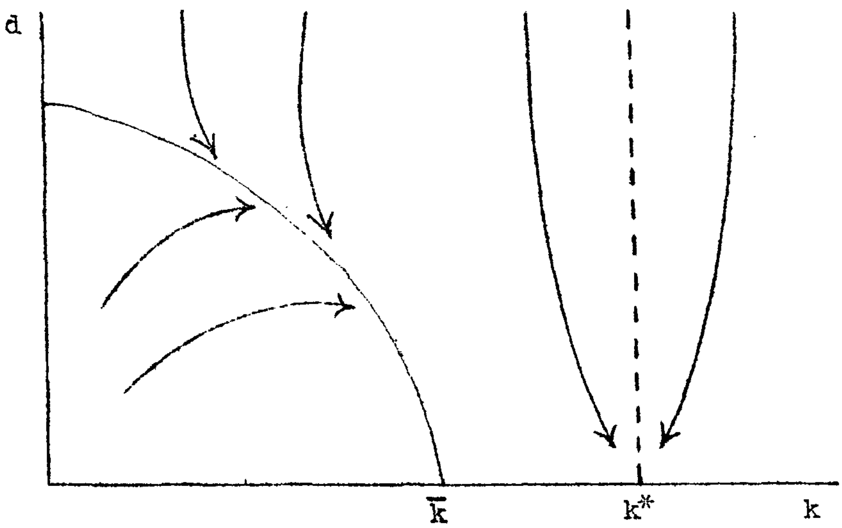
Now let us suppose that switching from I to II occurs. (in region A). Then to switch to the terminal phase of the optimal program there must be a switch to Pattern III. From Lemma 3 we know this can occur only in region B.

But once in Pattern II ($\sigma = \lambda$) and in region A $[f'(k)(1 - \alpha) > R'(d)]$ the economy cannot enter region B (i.e., $f'(k)(1 - \alpha) < R'(d)$), since debt (d) is decreasing. Hence, we have a contradiction.

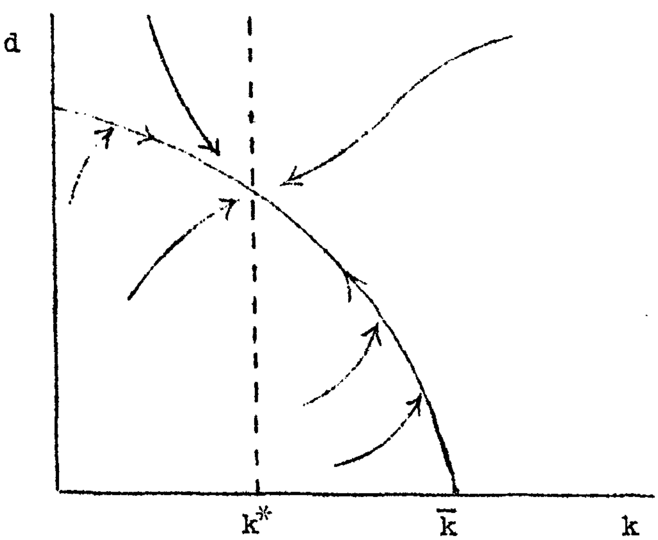
We can synthesize the controls in the following summary.

1. If the initial state of the economy is in region A, i.e., $f'(k)(1 - \alpha) > R'(d) + \mu$, then the optimal policy is $\sigma^* = 0$, i.e., accumulate debt and reach $d = g(k)$ as fast as possible and then switch over to the Euler trajectory.
2. If the initial state of the economy is in region B, i.e., $(1 - \alpha)f'(k) < R'(d) + \mu$ then the optimal policy is $\sigma^* = \lambda$, i.e., deaccumulate debt as fast as possible and when $d = g(k)$ switch over to the Euler trajectory.
3. If the economy is in region D, i.e., $k > k^*$, then deaccumulate capital as fast as possible and go to the stationary state.

The following phase-space diagrams depict the optimal program.



Case I $\bar{k} < k^*$



Case II $k^* < \bar{k}$

Further remarks. We conclude the analysis with several remarks.

1. The whole analysis can be easily extended if we allow neutral technical change in Harrod's sense (if the production function is Cobb-Douglas then Hicks' neutrality is the same as Harrod's neutrality). Such technical progress really means that we need a new measure of labor supply, i.e., "efficiency" units of labor. To be precise we have a production function

$$
\begin{aligned}
Y &= \tilde{F}(k,L,t) \\
&= F[k,A(t)L] \\
&= [A(t)L]F(k,1), \quad \text{where } k = k/[A(t)L] .
\end{aligned}
$$

Also,

$$y = \frac{Y}{A(t)L} = f(k) ,$$

so if $\frac{\dot{A}}{A} = b$ we can define the effective rate of increase in labor n' as $n' = n + b$ where n = the population growth rate. The entire preceding analysis can then be repeated.

2. If we allow the country to lend, then the only modification that occurs is that we have to additionally assume that the marginal interest earnings are a decreasing function of credit. We may then have in the stationary state $d < 0$, i.e., the country may become a creditor along the optimum path.

3. The terminal debt level depends upon $r(0)$ and δ. If δ is quite large then the country prefers to hold a debtor position rather than a creditor position.

4. The optimal debt level has another policy implication in the market economy. At this level the marginal productivity of capital is equal

to the marginal interest payments. Now if the borrowing is left to an atomistic borrower the borrowing tends to be excessive. The "myopic" private borrowing of such a borrower will fail to take account of the fact that an increase in his per capita borrowing increases the cost of borrowed capital for the society as a whole. He will borrow until the marginal productivity is equal to the average interest rate $r(d)$ rather than to $r'(d)$. This yields an inefficient program for the economy. There is then a clear case for an income tax on the earnings from the borrowed capital. One has to find the optimal tax rate t --the rate which will equate the private return to the social cost of borrowing.

CHAPTER IV

SOME NUMERICAL EXPERIMENTS IN THE CONTEXT OF THE INDIAN ECONOMY

A. Introduction

Professor S. Chakravarty [5], in a review of optimal investment planning models, notes that in these models the important question which is left unexplored is the sensitivity of the policy variables to the parameters of the model. Professor T. Koopmans [17], in his Irving Fisher Lecture to the Econometric Society, remarks that to date there is only one study available which explores the quantitative implications of the optimum growth models. The important parameters which determine the policy in these models are the time horizon, the terminal conditions, the utility function, and the discount rate. Interest in the time horizon and the terminal conditions is obvious. The planner would be naturally interested in the effect of "myopia" on the policy variables. Because of uncertainty about the future and information constraints, use of a shorter time horizon is appealing but may be sub-optimal. Similarly, the question of terminal conditions is both a logical and an ethical one. Any terminal condition is necessarily arbitrary and means specific capital resources bequeathed to future generations. Professors A. Manne and J. Barr [21] have explored these questions in their study of the one-sector, linear-technology growth model. They found that the policy variable (investment) is very sensitive to both the time horizon and the terminal conditions.

The utility function and the discount rates reflect the preferences of the economy. The assumption we have made about utility functions in our model (concavity) allows us to consider functions of the following type: $u(c) = c^{\eta}$ where $0 < \eta < 1$; η is the elasticity of utility and is assumed to be constant.

The utility function expresses the valuation of alternative levels of consumption. The marginal utility function $u'(c)$ represents the distributional weighting system for generations attaining different levels of consumption. On the optimum path $u'(c)$ is equated to the shadow price of capital (p_k). Thus with a concave utility function, $u'(c)$ operates much like a progressive tax system. But the weights implied by $u'(c)$ ignore the time at which the generation lives, and in the absence of any discount factor, the maximand, being an integral, tends to favor future consumption over current consumption in an economy satisfying our assumptions. Thus, due to the asymmetric nature of trade-off between the present and future generations the question of equity between generation arises. The discount rate δ reflects a judgment as to an equable distribution between generations.

We now report on some experimental studies in which there was special interest in the sensitivity of the policy variables to the valuation parameters (δ, η). The time profiles of the control variables generated in these experiments are not offered here as the policies that should be followed by the Indian Planners. Rather, the aim of the study is to make more explicit the relationships between the ethical values of the society (reflected by the parameters δ and η) and the optimal policies implied by these values.

If the terminal conditions on the state variables in the finite horizon model are the same as their stationary state values in the infinite-time problem, and if the horizon is long enough (i.e., the final state is attainable), then the optimal program is similar in a finite-horizon version to the optimal program in our original infinite-time version. So in these experiments the time horizon was fixed and was taken to be 50 years. The terminal conditions on the state variables were the corresponding stationary state values (k^*, d^*). The following values of δ and η were studied

$$\delta = .075,\ .1,\ .125,\ .15 \text{ and } \eta = .1,\ .2,\ .4,\ .6,\ .8$$

Thus, there were 20 cases in all.

B. <u>The Experimental Studies: Results</u>

The following are the conclusions that emerged from the numerical experiments.

1. In all the cases the investment program ($\dot{k}$) shows the turnpike property. In the beginning three years the rate of capital formation is quite high. The consumption in the beginning years in these programs turns out to be quite low. This may not be politically feasible and indeed, compared to present performance, the savings rate of .3 to .45 is quite unrealistic. There should be a minimum per capita constraint in these models, e.g. the utility function should be augmented with the constraint that $c(t) > \bar{c}$, all t, where $\bar{c}$ is a lower bound on per capita consumption.

This conclusion is further strengthened by the fact that, in underdeveloped economies, productive efficiency may be closely related to per capita consumption. Under-consumption may reduce productivity because of malnutrition, lack of public health services, etc.

2. The high capital formation rate in the first few years, even if the economy is willing, may be infeasible because of the lack of complementary inputs--skilled manpower, organizational skills, and so forth. To have realistic policy recommendations perhaps some ceiling on the investment rate should be included to reflect these constraints.

3. In the range of variation of δ and η explored in these experiments the values taken by the policy variables in the first few periods are much more sensitive to changes in η (the elasticity of utility with respect to consumption) than to changes in $\delta + \mu + n$. In fact, it turns out that if the value of $\delta + \mu + n$ is changed by x percent the initial values taken by the policy variable exhibit a change of the order of .05 x percent. This information is quite useful for the planner.

The term $\delta + \mu + n$ determines the terminal state to which the economy is driving itself. Now the coefficients μ and η are depreciation rate and population growth rate. These are empirical constants and difficult to measure. But suppose the planner has chosen $\delta = .10$. Then even if there is 50 percent error in the measurement of μ and n the initial response of the economy may be quite close to the true optimal policy had the correct information been available.

As we remarked in the last chapter, we have not included technical progress in our model. It turns out that if Harrod-neutral (autonomous) technical progress is present in the economy and if, say, its typical value is 1 to 1.5 percent per year, the initial response of the economy generated by our model would be close to the optimal response in a model incorporating such technical progress. For, as remarked in the preceding chapter, technical progress of θ percent per year only modifies our model by redefining the effective population growth rate at $n' = n + \theta$. Then $\delta + \mu + n'$ determines the terminal to which the economy is driven. But as we have noted, the sensitivity of the initial values of the policy variables to changes in θ is small. This is important for the planner as it is very difficult to measure the rate of technical progress.

Of course, our comments will not be true if technical progress stems from other sources, as, for example, from "learning by doing," as discussed by Arrow.

4. The exports required to keep the economy at the optimal debt level (i.e., the level where marginal interest payments equal marginal productivity of debt), given our estimated production function and interest rate schedule for the Indian Economy, turn out to be above 9 percent in the initial years. This is much above the present capacity of the Indian Economy to export.

Presently exports are about .06-.065 times G.N.P. This means that even if aid or borrowing is available, India should concentrate on exports expansion or else on import-substitution (i.e., on reducing β and α in our model). This also signals other underdeveloped economies that the available borrowing should not render them any sense of security. In fact, borrowing in the short run may involve in the long run, a serious problem of repayment. So it appears that expansion of trade and/or import-substitution in the capital goods industry is essential for the Indian economy. Of course, we have not taken into consideration the problems of worsening terms of trade, fluctuations in the exports earnings, the uncertainty of borrowing or aid agreements, and tied aid (i.e., the aid has to be used to purchase goods in the donor country). All these considerations may perhaps strengthen the arguments in favor of import-substitution and expansions of trade still further.

5. The results of the computations are summarized in the accompanying graphs. All the graphs depict the time-profile of the following variables: k, the capital/labor ratio; c, the per capita consumption; v, the consumption

coefficient; σ, per capita exports. Each graph depicts one of the twenty cases studied.

The horizontal axis is time and the vertical axis shows all the above-mentioned dependent variables. The number $k(0)$ represents the initial capital labor ratio. The base year (year 0) corresponds to the year 1964; the value of $k(0)$ is 1700 Rs per capita. The number $d(0)$ is the initial debt level if the debt is optimal (i.e., $(1 - \alpha)f'(k) - \mu = R'(d)$). Its value is determined, given the estimated production function and interest-rate schedule; its value turns out to be 151.3 Rs per capita. The actual value of per capita debt for the Indian economy in the year 1964 was 169 Rs per capita.

C. The Experimental Studies: Computational Techniques

The algorithm. The algorithm which was used to solve our problem was developed by NASA Research scientists at the Ames Research Center. We will briefly summarize it here. A fuller description is reported by E. Stewart et.al. [36]. The algorithm performs a global search for the optimal solution by solving the differential equations derived from the necessary conditions according to the Pontryagin Maximum Principle. In solving these equations one invariably encounters a mixed boundary-value problem. Usually initial conditions are known on the state variables and the transversality conditions give the end point conditions on the co-state-variables. So one has to find correct initial values of the co-state variables (the p_i's)--values such that the necessary conditions are satisfied by the resulting differential equations and so are the terminal conditions.

The algorithm used is called the "The Adaptive Random Search Algorithm". Its basic approach is to select the initial value of the vector of co-state variables from a "noise" source which changes from one repetition to the next, namely, a Gaussian distribution with the mean m^k and standard deviation σ^k where k is the number of the repetition. Thus, on the k-th repetition the vector of co-state variables is $p^k = m^k + z^k$ where $E(z^k) = 0$. In the k-th repetition p^k becomes $p(0)$, the initial value of the co-state vector. The values of $p(0)$ and $x(0)$ are sufficient to define a solution to the differential equation system

$$\dot{x} = f(x,u,t)$$
$$\dot{p} = g(x,u,t)$$
$$\dot{u} = u(x,p,t),$$

where

x = state vector

p = co-state vector

u = control vector,

and u maximizes the Hamiltonian H.

The final state $x^k(T)$ actually achieved (where T is the time horizon) will generally fail to satisfy the desired terminal conditions and similarly, the final values of the co-state variables $p^k(T)$ will fail to satisfy the transversality conditions. So a metric function is introduced to measure the difference between the desired terminal conditions and the actual boundary conditions.

The metric is the vector $J^k = (J_x^k, J_p^k)$ where J_x^k is itself a vector with coordinates $J_{x_i}^k$ corresponding to the state variables. $J_{x_i}^k$ takes one of two alternative forms, either $J_{x_i}{}^k = |x_i(T) - x_i{}^k(T)|$ or $J_{x_i}{}^k = \Sigma[x_i(T) - x_i{}^k(T)]^2$, where $x_i(T)$ is the desired terminal value of the state variable x_i and $x_i{}^k(T)$ is the actual terminal value at the k-th repetition. The vector $J_p{}^k$ has coordinates which take one of the two analogous forms for each of the co-state variables. The correct value of p(0) will make the value of J equal to zero. But in the actual computer simulation the boundary conditions on the x and p will never be precisely achieved. Hence, an arbitrary vector ε is chosen such that the terminal set is enlarged until the metric J is less than ε.

The purely random search described above has long convergence time so an adaptive random search is performed by the algorithm. This improves the convergence property by varying the mean m and the standard deviation σ as a function of the past performance, as measured by the metric J. To be precise,

$$m^{k+1} = m(z^1, z^2, \ldots, z^k \quad J^0, J^1, \ldots, J^k)$$

$$\sigma^{k+1} = \sigma(z^1, z^2, \ldots, z^k \quad J^0, J^1, \ldots, J^k)$$

$$\text{and } p^{k+1} = m^{k+1} + z^{k+1} .$$

Also,

$$E(z^{k+1}) = 0$$

$$\mathrm{Cov}(z^j, z^k) = \begin{cases} Q & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

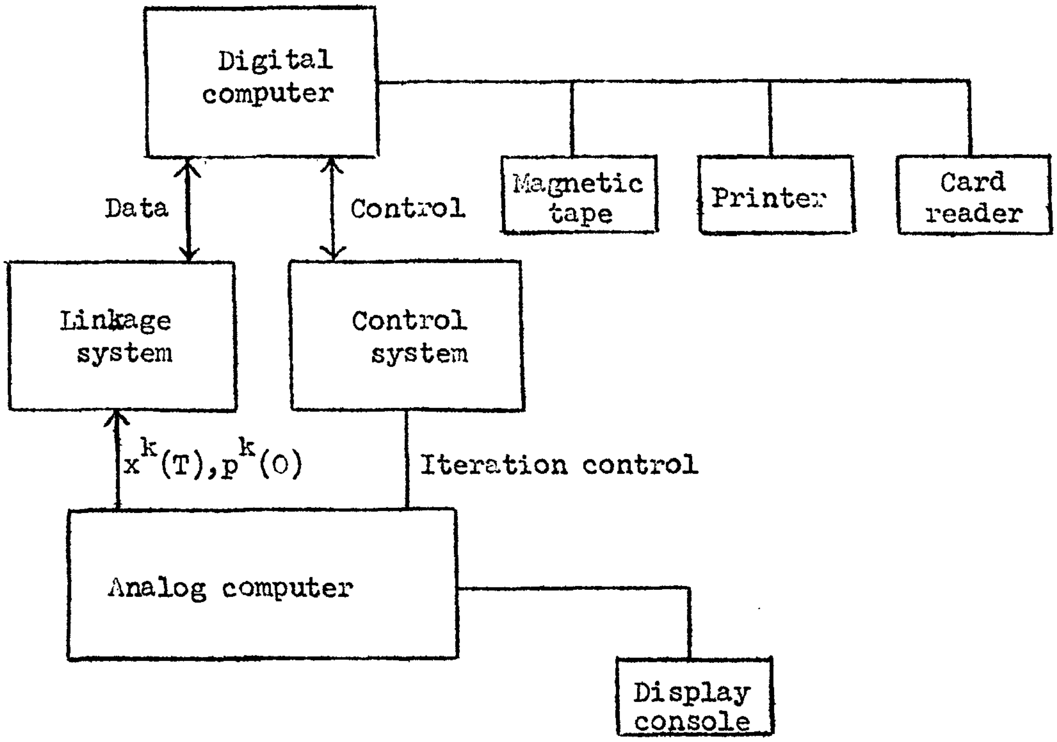
<u>Implementation</u>. The hybrid computer (digital-analog) system was used to implement the program. Each repetition can be divided into the two steps 1) integration of the equations of the motion in the interval [0,T] and 2) execution of the algorithm. The analog computer does integration in Step 1 10^3 times faster than the digital computer while the second step is done best by the digital computer.

The analog, however, is not as accurate as the digital. In solving our problem we found that the initial value of p has to be correct up to 5th digit accuracy on each of the coordinates. This restricts the use of the analog computer. But, in our problem we could implement the program to investigate the trajectory, provided the economy has optimal debt level initially. For in this case on the optimum trajectory we have $p_1 = p_2$, which implies that we have to guess only one value of p. This was then within the accuracy of the analog computer.

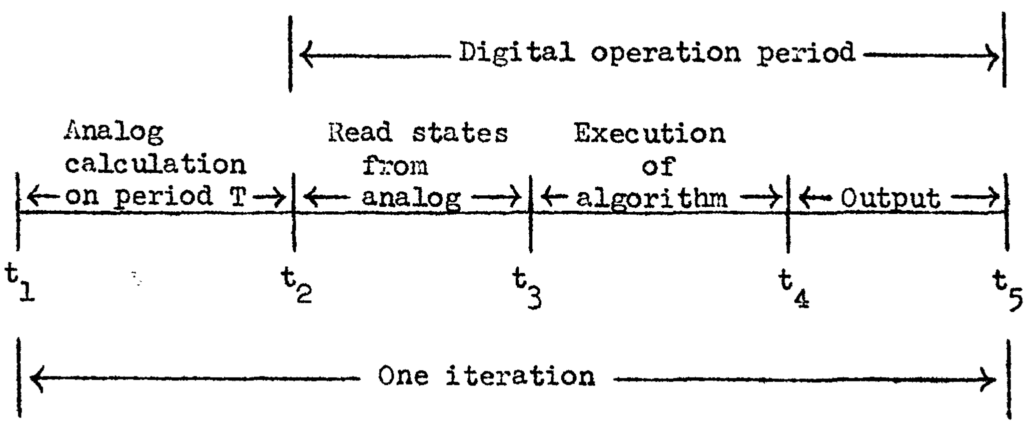
The digital computer used in executing this program was an Electronic Associates, Inc. (EAI) model 8400. It is a medium size high-speed computer with 16,000 words of core memory with 32 bits per word.

The analog hardware consisted of an EAI 231 R-V analog computer. The state equations and co-state equations were programmed in standard fashion. For continuous output a display console was connected to the analog computer to provide visual read-out of the variables.

The following diagrams show the sequence of events during one repetition and the hybrid system hardware.



Hybrid System Hardware



D. Estimation of the Parameters used in the Experimental Studies

1. The Production Function

A production function is a technological relationship. It specifies the maximum possible output for given input combinations. If the outputs are many, then we may have to use a set-theoretic concept of production possibility set, i.e., a point-set mapping.

The aggregate production function is used extensively in modern economic growth literature. The criticism against using such a concept is fierce. Critics argue that there is no such thing as an aggregate capital stock (i.e., capital "jelly") and, that, in fact, the economy has heterogeneous capital goods and aggregation is almost impossible (unless there are extreme restrictive conditions; see Green [ ]). Critics of the aggregate production function recommend the use of activity analysis, which describes reality better than the neo-classical production function.

The use of the aggregate production function can be defended on the following grounds: 1) the enormous data and computational requirements of handling the disaggregated capital goods model; 2) the use of the aggregate production function in the "surrogate" sense. The argument here is that the economy behaves as if it has such a production function on the macro-level where there are homogeneous inputs; 3) in much of the development planning literature, the simple fixed-proportion Harrod-Domar type production function is used. Use of such a function completely ignores the substitution possibilities for underdeveloped economies. Professor B. Hansen has shown persuasively, from empirical findings, that such phenomena exist in the U.A.R. [ ]. Professor Schultze has similar observations on many Latin American economies[ ].

We will argue that the neo-classical production function is a better approximation than the ones chiefly used so far in the development planning literature.

The assumptions. We will assume that there exists a relationship on the macro-level between aggregate output and inputs. We will assume that the production function is of Cobb-Douglas form with constant returns to scale. We will assume that there is no technical progress, embodied or disembodied,

$$\text{i.e.,} \quad Y = AK^{\alpha} L^{1-\alpha} \quad \text{where } 0 < \alpha < 1;\ A > 0$$

$$\text{or} \quad y = Ak^{\alpha} \quad \text{where } y = \frac{Y}{L}, \quad k = \frac{K}{L}$$

There are cross-section studies available which estimate a production function for India's industry and agriculture [4]. The production functions estimated were also of the Cobb-Douglas variety. The study showed that the sum of the capital and labor elasticities is very close to one for industry. The agricultural studies were for individual crops and the sum of the elasticities varied from crop to crop. For all crops, however, the values of the sum of the elasticities were close to 1 (between .9 and .95). Land was included in the agricultural studies as a capital stock. It seems, then, that there is no significant indication of the presence of any sharp decreasing returns to scale or increasing returns to scale.

The assumption of the absence of autonomous technical change or progress is admittedly a weakness. However, there are many statistical and identification problems in estimating such a phenomenon. The data is rather scanty and in many cases, not available. A simple time trend estimate of autonomous technical progress from time series data on aggregates creates the problem of colinearity in the estimates. However, there is an argument in our favor. The study done by Professors D. Jorgenson and Z. Gvilliches [15] shows that, at least for the U.S., there is no evidence of the existence of autonomous technical progress, i.e., they show that almost all the increase in the output can be explained by the increase in the inputs. Arguments on both sides are being exchanged and the issue is far from settled.

The data that is used here are time series data for aggregate output, capital stock and labor.

Output data. We will use the gross national product (G.N.P.) as the indicator of the total output. The time series is in constant prices. The data are published by the Central Statistical Organization of the Government of India. The published data are, however, for national income so we have constructed G.N.P. data from the gross and net investments estimates published by the Reserve Bank of India (G.N.P. equals national income + (gross - net) investments). A methodological question that arises is how to treat the output of the public entarprises. In India, this is quite a sizable sector, especially in the manufacturing goods industries. We have opted to include the public enterprises output in the G.N.P. time series. The time series are shown in Table 1.

Labor services. This variable should be measured in terms of a standardized unit, e.g., equivalent man hours. There are two problems in generating such data: one has to aggregate the different kinds of labor endowed with different skills and having different demographic characteristics (age, sex, etc.); and the weights used in the aggregation may have to change over time if the aggregation is to be meaningful.

For India the data availability is rather scanty. The only data available are those of the "economically active population" (i.e., the population between age 15-50 years) from the census studies in 1951 and 1961. There are data available on "labor participation" which provide some estimates of under-

Table 1

(Data are in billion Rs and constant 1948-49 prices)

Year	[1]National income	[2]Gross I_t	[2]Net I_t	G.N.P.
51	88.50	10.06	7.21	90.35
52	91.00	11.57	8.17	94.20
53	94.60	10.93	7.84	98.71
54	100.00	11.25	8.09	103.46
55	102.80	13.96	10.29	106.47
56	104.80	16.46	12.61	108.65
57	110.00	18.74	14.46	114.28
58	109.00	21.75	16.78	114.00
59	116.50	18.13	13.17	120.50
60	118.60	20.17	14.81	124.93
61	127.30	24.71	18.26	133.70
62	130.60	24.71	17.35	138.00
63	133.00	26.78	18.75	141.00
64	139.00	25.80	16.92	147.50
65	150.00	26.00	16.46	159.00

[1]Source: "Statistical Abstract of the Indian Union," Central Statistical Organization, Government of India, New Delhi, 1966.

[2]Source: "Savings and Investment in India," N.C.A.E.R., New Delhi, 1966.

employment and unemployment. For the estimates of labor inputs in non-census years we make appropriate assumptions about the labor growth rates. We assume the constancy of the demographic structure of the population and the rate of utilization of labor. In the post-census years we assume the growth rate of labor to be 2.25. This is the growth rate postulated by the planning commission. The resulting estimates are shown in Table 2.

Table 2

Labor Estimates*
Economically active population (1961) (in millions)

	Total	Econ. Active	
Male	226.29	129.17	
Female	212.9	59.5	
	439.19	188.67	Totals

Year	Labor
51	134.4
52	137.4
53	140.3
54	143.8
55	147.0
56	150.1
57	153.4
58	157.0
59	162.3
60	168.0
61	172.0
62	175.6
63	179.4
64	183.0
65	

*Source: "Statistical Abstract of the Indian Union, Central Statistical Organization, Government of India, New Delhi, 1966.

Capital-stock data. Capital stock is more difficult to measure. In an ideal world where all the machines are alike at each instant of time, the ideal measure would be the number of homogeneous machines. But, in reality, capital stock consists of various kinds of machines, buildings and lands, and each may be of a different vintage. The only way one can aggregate then is in value terms--i.e., in some constant Rupee value. The ideal price index to use in obtaining constant-price estimates would reflect the changing values of capital services. Since such an index was not available we used the investment goods price index (shown in Table 3) as a deflator. Because of the paucity of data on the age structure of the capital stock we also approximated capital services by using the net capital stock instead.

Table 3

Price Index (1952-53-100)*

Year	General price index	Investment price index
51	125.0	119.0
56	102.7	104.9
57	108.7	108.0
58	111.0	108.0
59	115.5	110.0
60	130.0	121.0
61	125.8	127.2
62	127.5	128.1
63	132.5	130.0
64	148.3	135.0

*Source: Reserve Bank of India Bulletin, Bombay, 1965.

There are no capital stock time series available from the government publications. The only data available are the investment time series estimated by the Reserve Bank of India. [31] So the capital stock for at least one year has to be constructed and then we can generate the capital stock series from the investment time series.

The year 1960-61 was chosen as the base year because data were available on agriculturallcapital stock in that year. This was due to a painstaking study by Professor Tara Shukla [34]. The data were gathered by Perspective Planning Division of the Planning Commission for the manufacturing services, transportation, communications, and other sectors for the year 1960-61. There was a separate study by the Reserve Bank of India measuring the total tangible wealth in India. Both the above studies were quite close to each other. The agricultural and other capital stocks were aggregated for that year. The resulting series is shown in Table 4.

Table 4

Capital Stock in Billion Rs and in Constant Prices (1948-49)

Year	Capital stock
51	152.1
52	156.3
53	162.7
54	168.5
55	174.5
56	182.8
57	193.3
58	205.9
59	220.7
60	232.3
61	245.6
62	282.0
63	300.0
64	317.4
65	333.0

<u>Estimation results</u>. The total number of observations available to estimate the production function was 15. However, the observation for the year 1958 was dropped because of the unusual drought in that year, which depressed the total output. The index of capacity utilization used was .9, estimated to be a reasonable approximation; there were evidences of excess capacity in some sectors. The equation estimated was

$$\log Y = a + b \log k \qquad \text{where } y = \frac{Y}{L}$$

$$k = \frac{K}{L}$$

The regression gave the following estimates of a and b (standard errors in parentheses):

$$\hat{a} = 120 \ (.246109),$$

$$\hat{b} = .3445 \ (.034084)$$

$$R^2 = .8871$$

The estimated production function is

$$y = 120k^{.3445}$$

$$Y = 120k^{.3445} \ L^{.6555}$$

If the markets are perfectly competitive and there is profit maximization then the shares of wages and profits equal their respective output elasticities. The capital markets in the Indian economy are far from being perfect and in many sectors there is monopoly. So we cannot expect the available data on the shares of income to be consistent with the output elasticity of our estimate. However, in Table 5 we reproduce the available data to compare these figures.

Table 5*

Distribution of Income (the Whole Economy)
(in current prices and in billion Rs)

Year	Wages and salaries	Profit + rent + interest	Income from self-employment	Total income
49	24.33	18.29	44.05	86.66
50	25.06	19.08	46.04	90.37
51	26.08	20.76	48.56	95.40
52	27.50	22.14	50.08	99.72
53	27.81	21.68	48.79	98.23
54	29.28	23.35	52.12	104.76
55	29.24	22.52	44.32	96.08
56	30.32	23.40	46.00	99.80
57	33.00	27.00	53.00	113.00
58	34.80	27.90	51.20	114.00

*Source: Balbir, Saham, "Savings and Economic Development," Calcutta, 1967, Table V.2, p. 63.

The "income from self-employment" comes from the services sectors and some of the agricultural sectors. It is possible that in these sectors the wage share may be higher as a percentage of total product than in the remaining sectors because of its labor-intensive nature. Taking account of this possibility, a reasonable guess is that the aggregate income share of capital is between .3 and .4. This is close to the elasticity of capital in our estimated production function. This is not a proof that the true function has the form assumed, or that our estimated coefficients are reliable. But it gives some additional confidence in the estimates.

2. Rate of Depreciation

The nature of depreciation is assumed to be radioactive (i.e., exponential). Several types of capital goods have different rates of depreciation. The study of Professor A. Manne [22] of the 4th five-year plan gives estimates of these rates. Similarly, Tendulkar's work [37] gives some estimates. We chose the average of all these estimates. This comes out to be .035 (i.e., μ = .035, in the terminology of our model).

3. Maintenance Imports

These imports are related to the goods imported to sustain current economic activities. The imports are of several types: industrial raw materials, fuels, spare parts, etc. The study made by NCAER (National Council for Applied Economics Research) on "maintenance imports" was the basis for the estimation of our study [24]. The time series is shown in Table 6.

Table 6*

(In billion Rs)

Year	Maintenance imports	Total imports
52	4.59	9.70
53	3.00	6.67
54	2.91	5.67
55	3.27	6.56
56	3.60	6.83
57	4.62	8.58
58	5.01	10.35
59	3.68	8.59
60	4.49	9.60
61	5.119	11.21
62	5.00	10.38
63	4.91	11.31
64	4.872	11.43
65	N.A.	N.A.

*Source: N.C.A.E.R., "Maintenance Imports," New Delhi, 1967.

The average value of α turns out to be equal to .02 where maintenance imports = αY and where Y = total output.

4. Investment Imports

These imports are of the nature of externally produced investment goods, e.g., machine tools, fertilizer plant, etc. They are complementary to the domestic investments. The planning commission's draft outline of the 3rd five-year plan (1961-66) and the 4th five-year plan (1967-72) were used as the basis for the estimation of the coefficient β, as shown in Table 7.

Table 7

Investment and Foreign Exchange Requirements (in billion Rs)

	Total outlay		Foreign exchange requirement	
	3rd plan[1]	4th plan[2]	3rd plan	4th plan
Public sector	61.00	N.A.	15.20	N.A.
Private sector	43.00	N.A.	5.10	N.A.
Totals	124.00	237.50	20.30	38.50

Estimate: $\beta = .2$

[1]Source: <u>Third Five-year Plan</u>, Planning Commission, Government of India.

[2]Source: <u>Fourth Five-year Plan</u>, Planning Commission, Table II and Table V, Government of India.

5. Exports

The data we have are the observed levels of the exports and what we really want to estimate is λ, the maximum propensity to export. In our judgment, the economy, whose planners have shown considerable attention to the scarcity of foreign exchange, will export as much as possible. This may especially be true when an important criterion used by foreign-aid donors in providing external lending is the performance of the exports of the domestic economy. The observed exports will give a reasonable estimate of λ.

Table 8*

Year	Exports (billion Rs)	Exports (percentage of G.N.P.)
55-56	6.09	6.0
58-59	6.20	5.0
60-61	6.50	4.5
61-62	6.70	4.5
62-63	6.85	4.7
63-64	7.90	4.7
64-65	8.10	4.8

*Source: "Statistical Abstract of the Indian Union," Government of India, New Delhi, 1967.

This table indicates that λ is in the range .05 to .075.

6. Debt

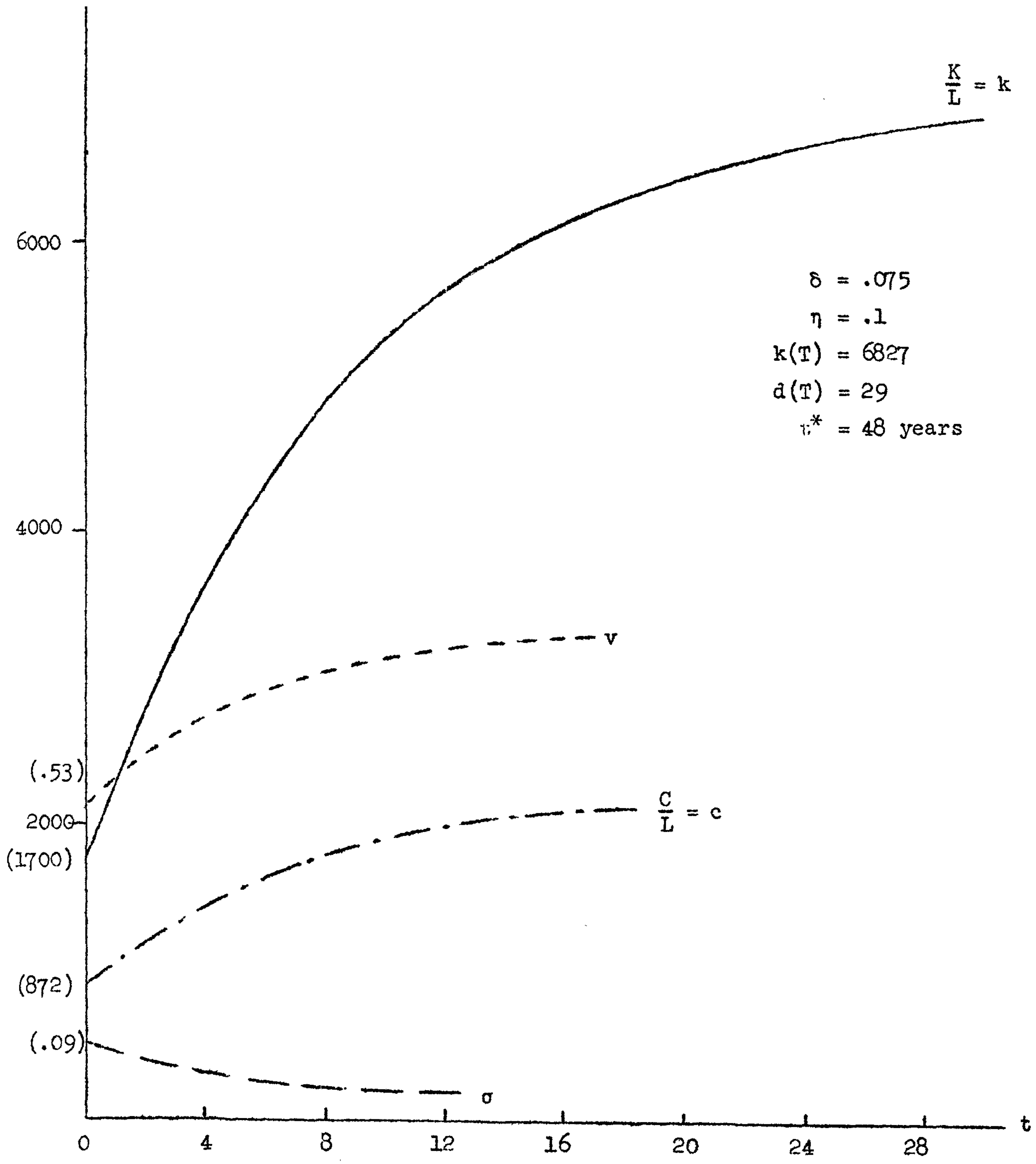
The Ministry of France of the Government of India publishes data on external assistance. These figures give the data for the years 1951-1964 on the amount of resources allotted, the donor, the terms of loans. The data are first transformed into Rupees from the different currencies (dollars, pounds, marks, etc.) using the official exchange rates. Some loans were not fully utilized. Unfortunately, the data do not give exactly by how much any particular loan was utilized. So, to construct the interest rate schedule the following assumption was made: the actual composition of the utilization of the loans bearing different interest rates is the same as their composition in the total loans allotted. The following table summarizes the data.

Table 9

Interest rate	Million Rs	Cumulative
4.0%	1467	1467
5.0	2725	4192
5.5	6498	10690
6.0	7950	18640
6.5	1725	20365

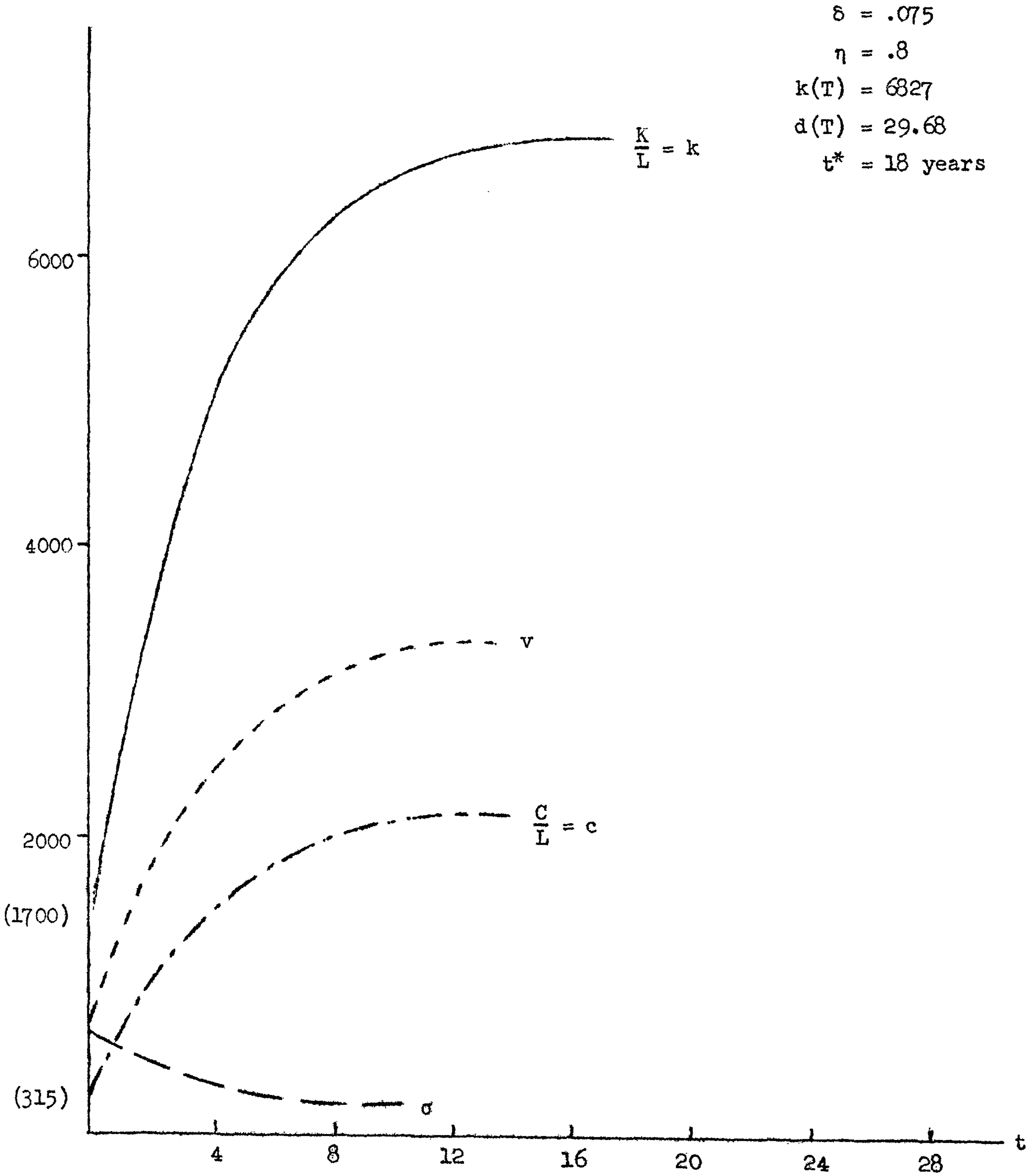
The number of observations are rather few. The following interest rate schedule function was used. The independent variable is the per capita debt (d) and the dependent variable is the interest rate r(d).

$$r(d) = .05 + .0008\ d.$$

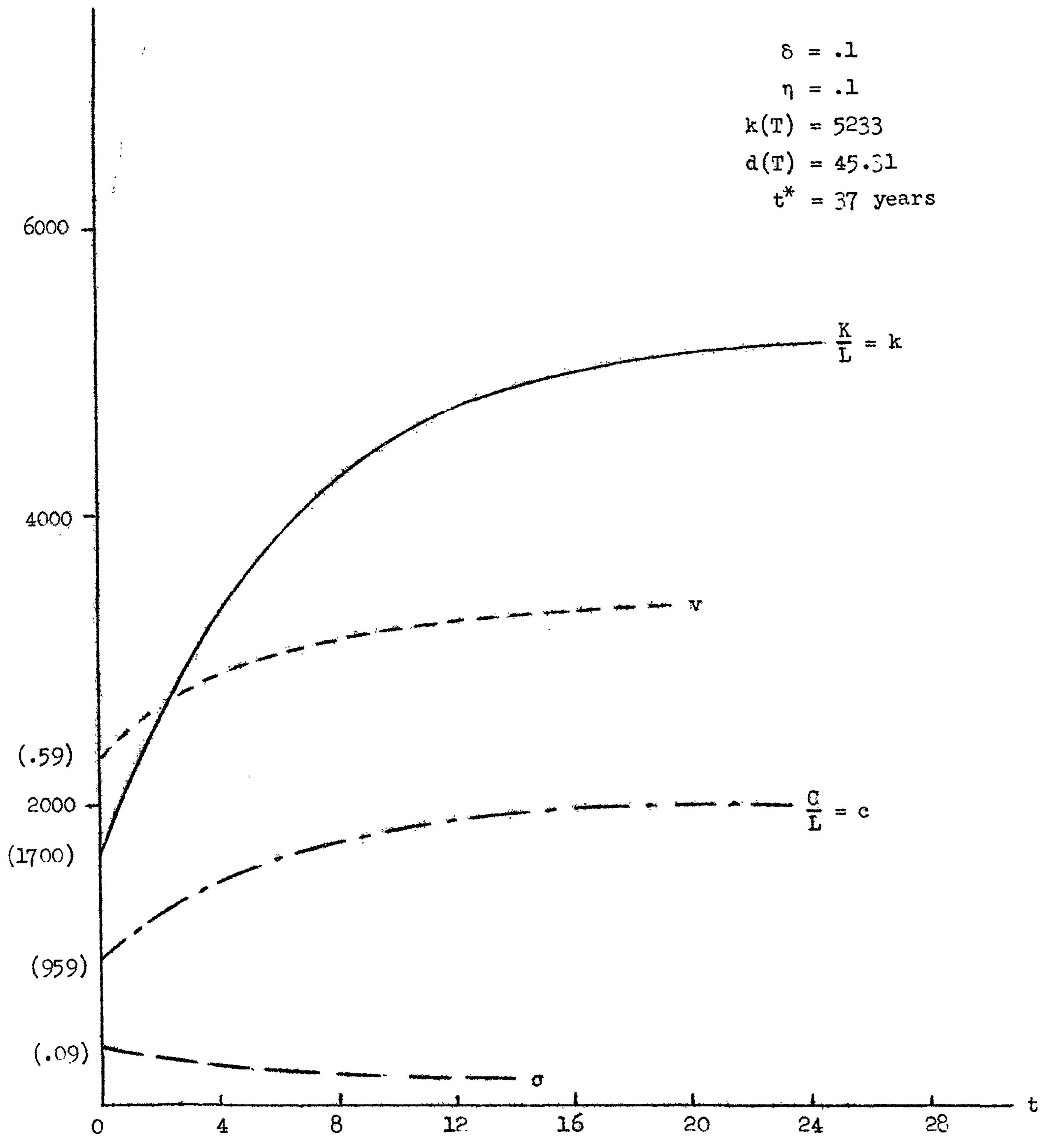


$\frac{K}{L} = k$
6000
4000
2000
(.53)
(1700)
(872)
(.09)
$\delta = .075$
$\eta = .1$
$k(T) = 6827$
$d(T) = 29$
$v^* = 48$ years
v
$\frac{C}{L} = c$
σ
0
4
8
12
16
20
24
28
t

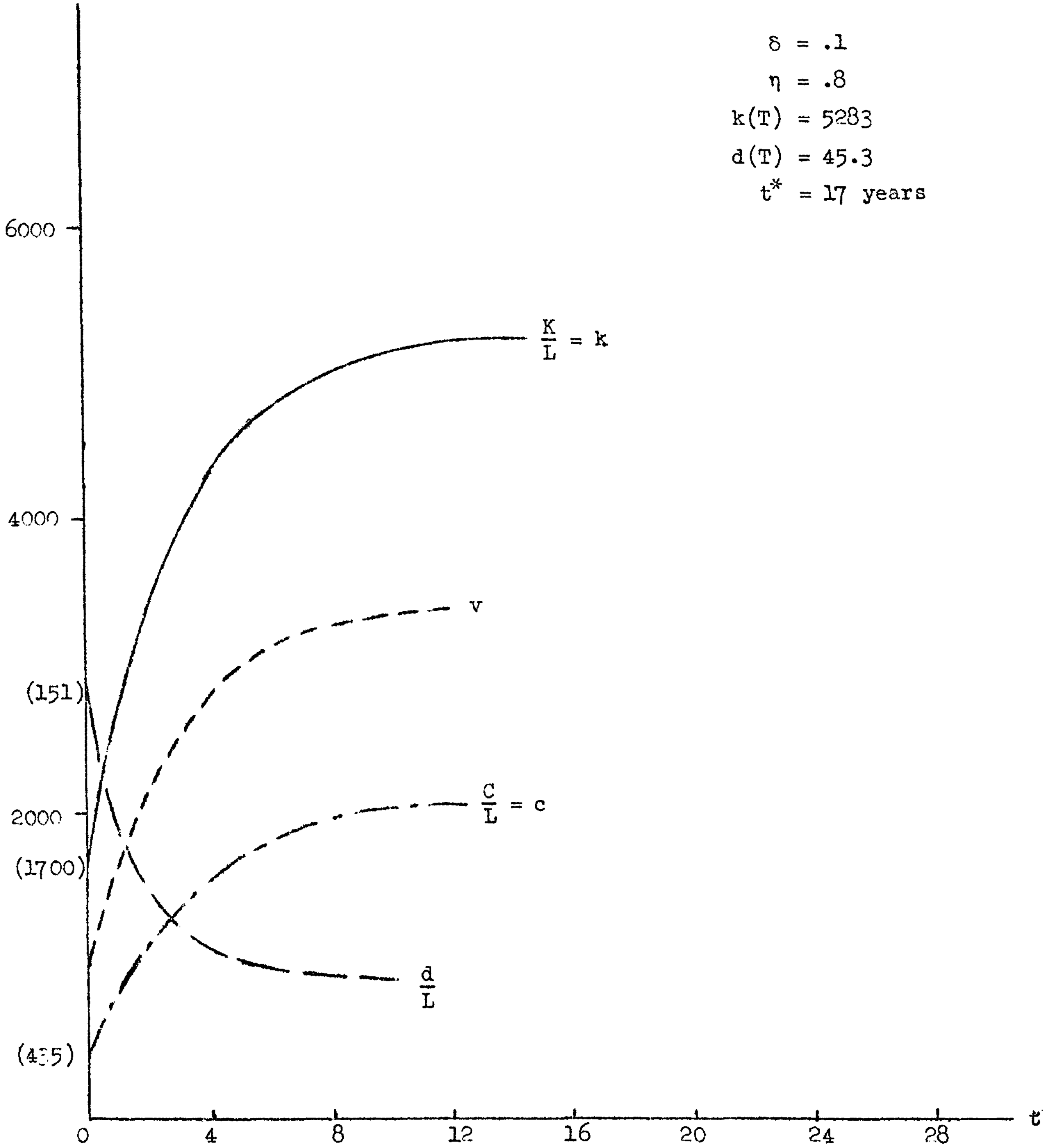
Graph 1


δ = .075
η = .8
k(T) = 6827
d(T) = 29.68
t* = 18 years
K/L = k
v
C/L = c
σ
6000
4000
2000
(1700)
(315)
4
8
12
16
20
24
28
t

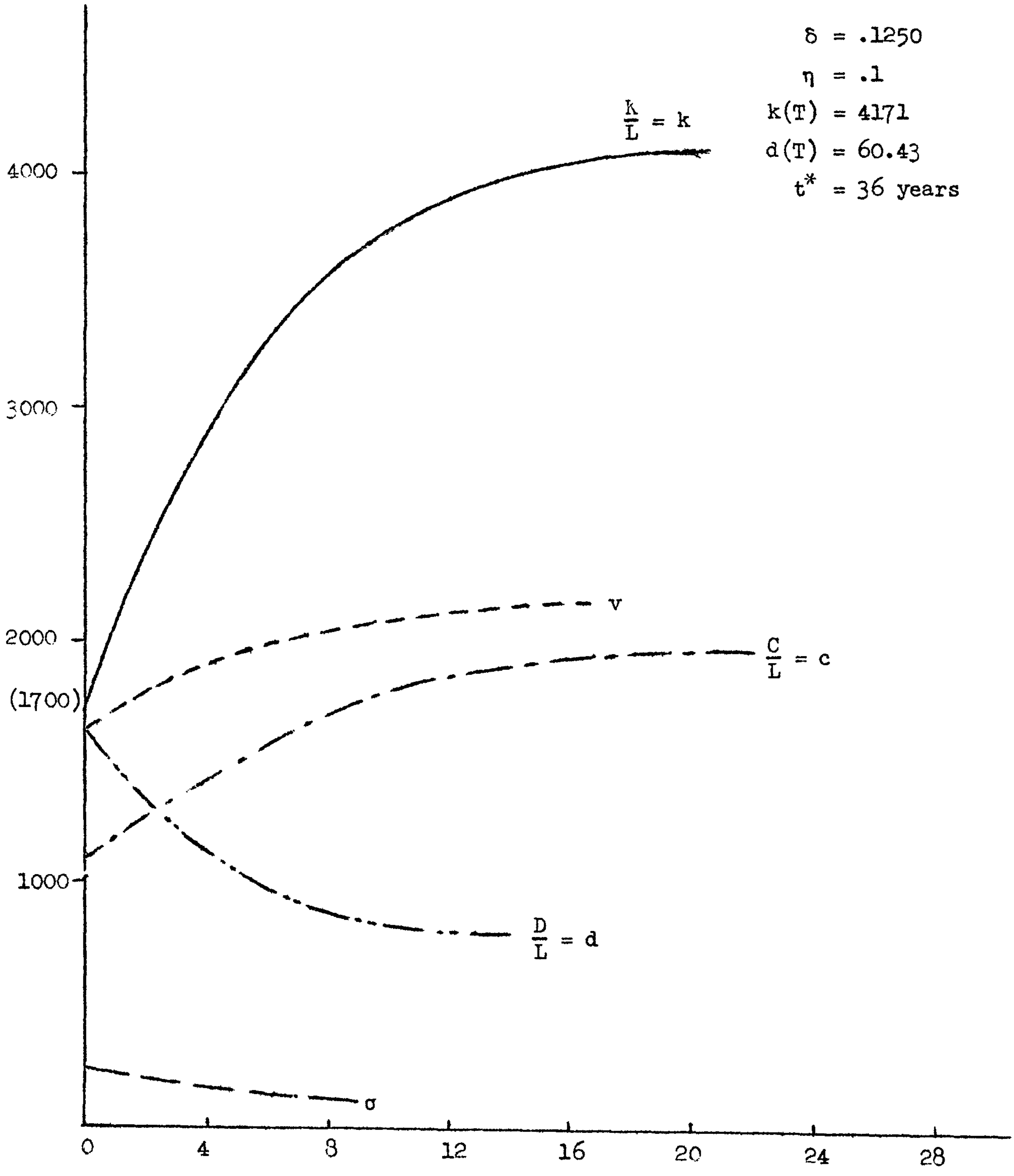
Graph 2


δ = .1
η = .1
k(T) = 5233
d(T) = 45.31
t* = 37 years
6000
4000
(.59)
2000
(1700)
(959)
(.09)
K/L = k
v
C/L = c
σ
0
4
8
12
16
20
24
28
t

Graph 3

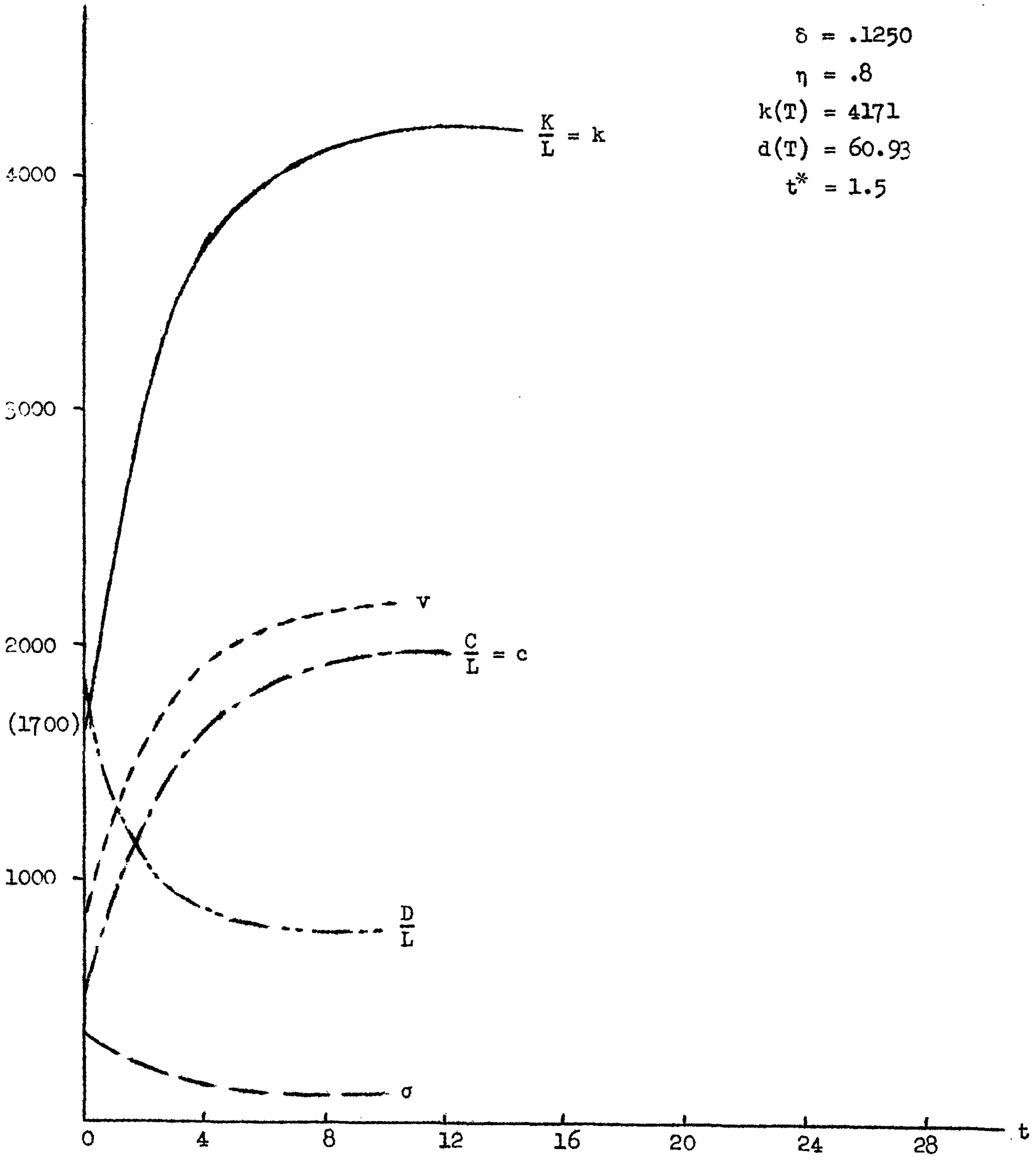

δ = .1
η = .8
k(T) = 5283
d(T) = 45.3
t* = 17 years
6000
4000
(151)
2000
(1700)
(435)
K/L = k
v
C/L = c
d/L
0
4
8
12
16
20
24
28
t

Graph 4

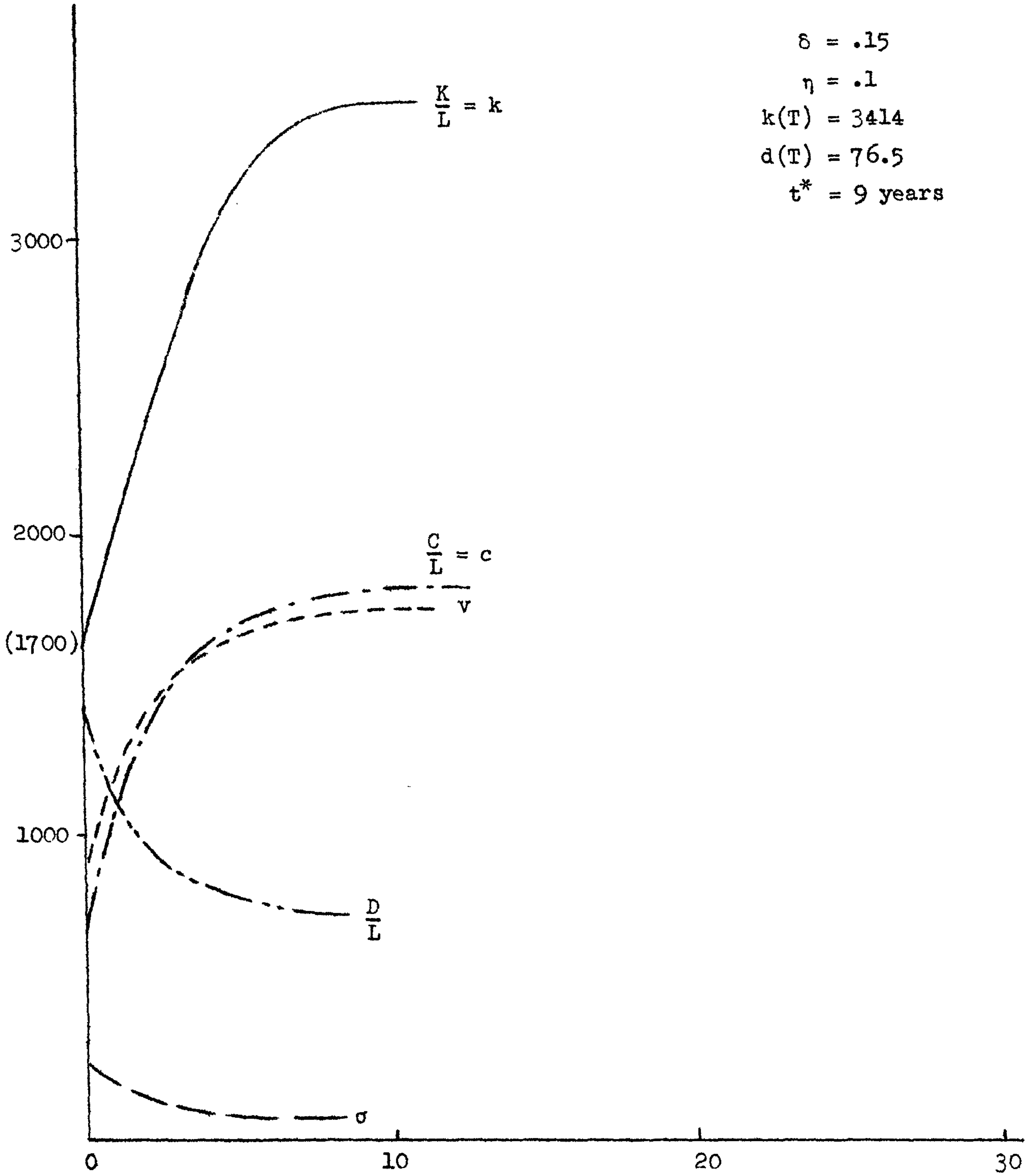


δ = .1250
η = .1
k(T) = 4171
d(T) = 60.43
t* = 36 years
K/L = k
v
C/L = c
D/L = d
σ
4000
3000
2000
(1700)
1000
0
4
8
12
16
20
24
28

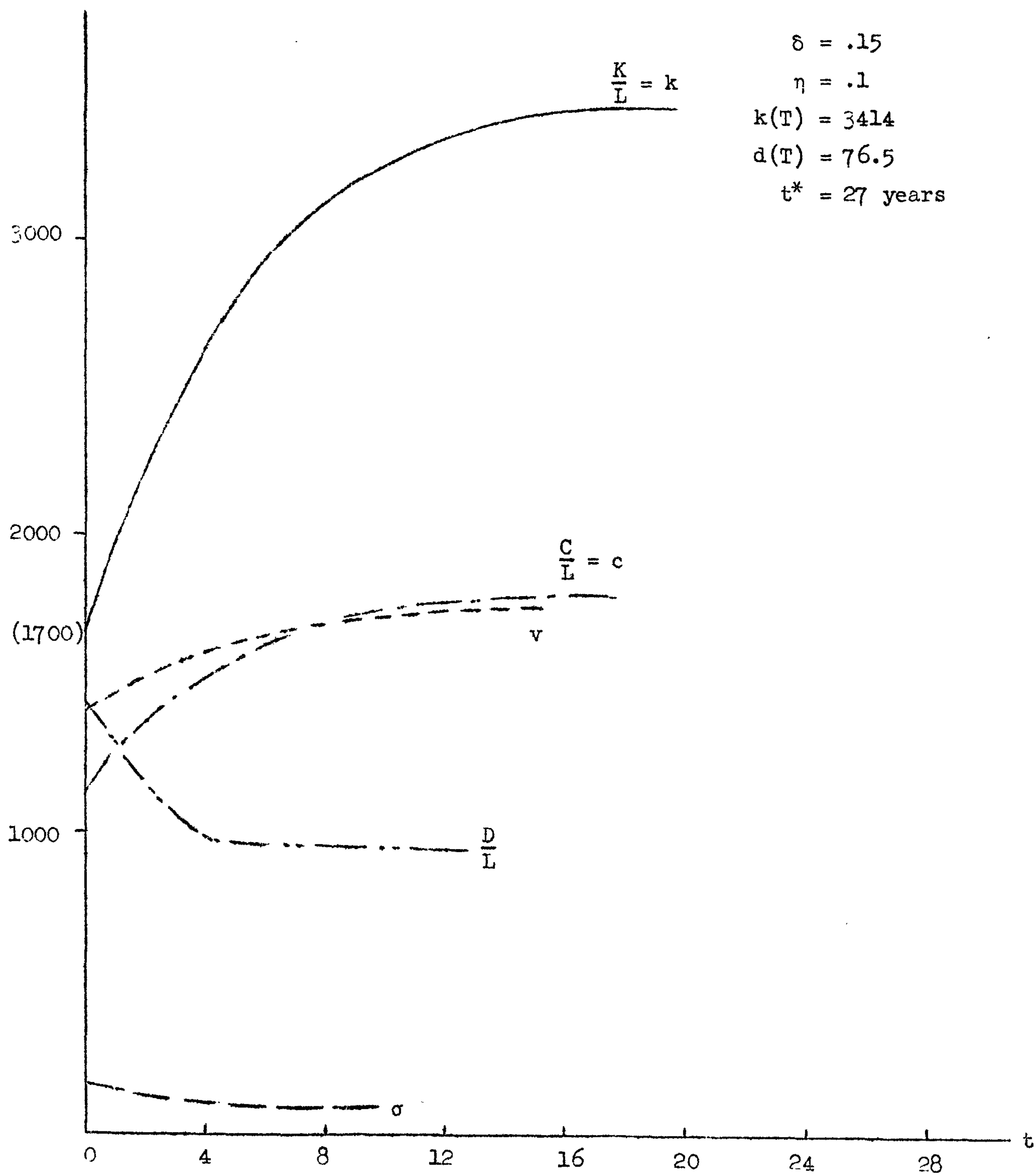
Graph 5


δ = .1250
η = .8
k(T) = 4171
d(T) = 60.93
t* = 1.5
K/L = k
v
C/L = c
D/L
σ
4000
3000
2000
(1700)
1000
0
4
8
12
16
20
24
28
t

Graph 6


δ = .15
η = .1
k(T) = 3414
d(T) = 76.5
t* = 9 years
K/L = k
C/L = c
v
D/L
σ
3000
2000
(1700)
1000
0
10
20
30

Graph 7



δ = .15
η = .1
k(T) = 3414
d(T) = 76.5
t* = 27 years
K/L = k
C/L = c
v
D/L
σ
3000
2000
(1700)
1000
0
4
8
12
16
20
24
28
t

Graph 8

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